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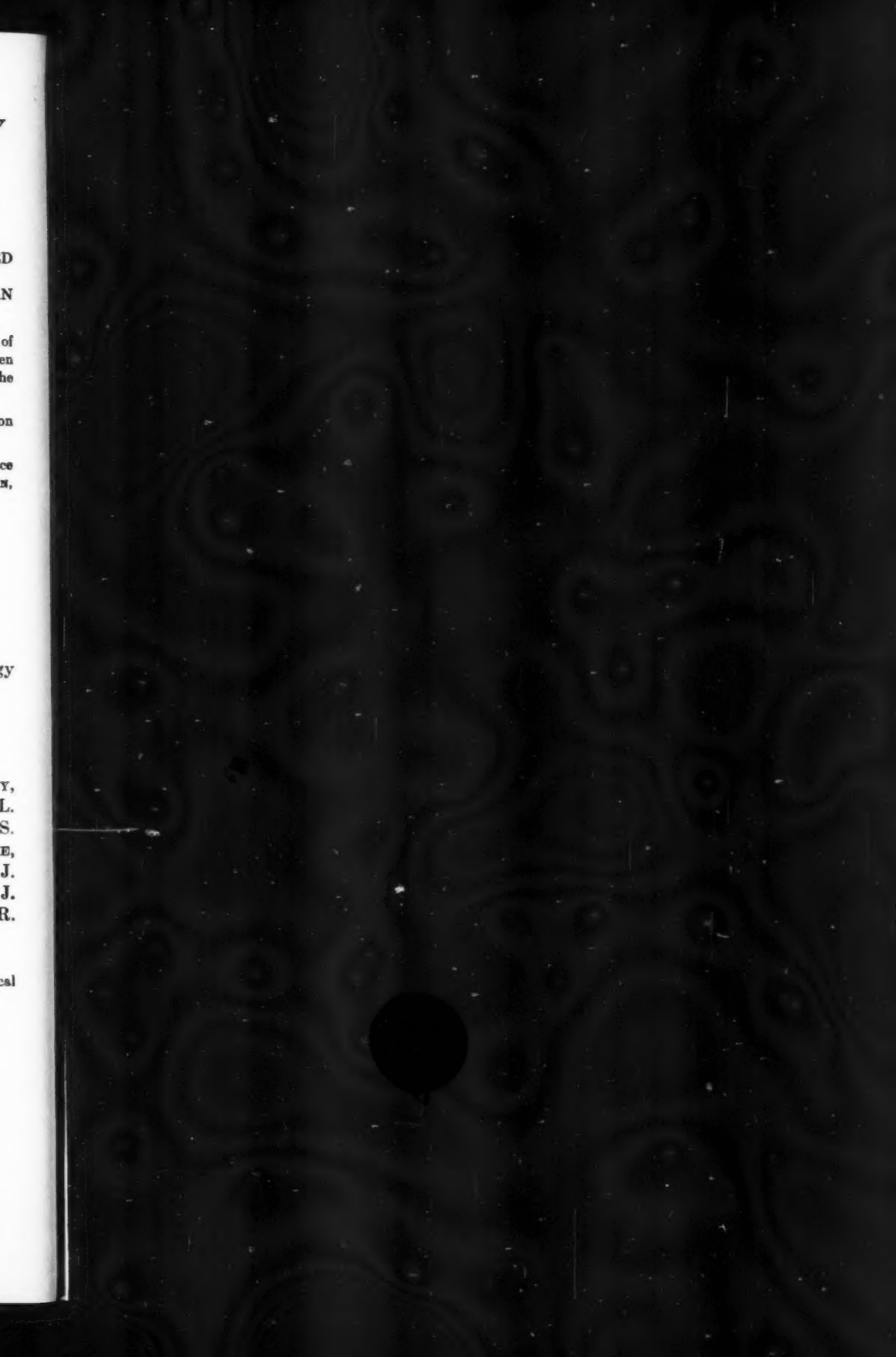

A BUDGET OF CURIOSA METRICA

LEONARD M. BLUMENTHAL, University of Missouri

Nearly twenty years ago, this MONTHLY published an article by the writer devoted to those remarkable mathematical paradoxes due to Sierpiński, Mazurkiewicz, Hausdorff, Banach, and Tarski [4]. No such startling results are contained in the present paper, the title of which was obviously suggested by De Morgan's famous *Budget*, but perhaps the reader with sufficient good will might agree that at least some of its material explains (if it does not excuse) the use of the term *curiosa*.

1. A new inequality for tetrahedra. Let 1, 2, 3, 4, denote the vertices of a euclidean tetrahedron, perhaps degenerate, and e_i the sum of the two smallest angles of that face (triangle) of the tetrahedron which is opposite vertex i, $(i=1, 2, 3, 4)$. Pauc calls e_i the flatness (*aplatissement*) of the triangle, and his *Thesis* contains a theorem which is said to follow from a lemma that asserts (without proof) the inequality

$$e_i + e_j + e_k \geqq e_m,$$

where (i, j, k, m) is any permutation of $(1, 2, 3, 4)$ [10].

Very recently Pauc mentioned to the writer that a proof of the inequality was still lacking. In discussions with my student, C. J. Penning, the inequality was established. The following is an adaptation of his argument.

PENNING-PAUC INEQUALITY. *If* 1, 2, 3, 4 *denote four points of a euclidean space, the sum of any three of the four numbers* e_i $(i=1, 2, 3, 4)$ *is greater than or equal to the fourth number, where* e_i *denotes the sum of the two smallest angles of the triangle whose vertices are obtained from* 1, 2, 3, 4 *by omitting point* i, $(i=1, 2, 3, 4)$.

Proof. If ϵ_i denotes a largest angle in the triangle whose flatness is e_i, then the inequality to be established is equivalent to

$$(*) \qquad \epsilon_i + \epsilon_j + \epsilon_k - \epsilon_m \leqq 2\pi,$$

for every permutation (i, j, k, m) of $(1, 2, 3, 4)$.

Case 1. *Two largest edges of the tetrahedron are opposite.* Assume the labelling of the vertices so that those two edges are (1, 2) and (3, 4), and develop the tetrahedron on the plane of the face (1, 2, 3). The planar figure so obtained is formed by the triangle $\Delta(1, 2, 3)$ and the three triangles $\Delta(1, 2, 4_3)$, $\Delta(2, 3, 4_1)$, $\Delta(1, 3, 4_2)$ constructed on its sides by folding "outwards" the three triangular faces of the tetrahedron that meet at vertex 4.

The assumption characterizing Case 1 permits the identification of each of the angles ϵ_n $(n=1, 2, 3, 4)$, for it is clear that $\epsilon_1 = \max \angle(2, 3, 4) = \angle 2: 3, 4_1$, $\epsilon_2 = \max \angle(1, 3, 4) = \angle 1: 3, 4_2$, $\epsilon_3 = \max \angle(1, 2, 4) = \angle 4_3: 1, 2$, $\epsilon_4 = \max \angle(1, 2, 3) = \angle 3: 1, 2$, where $\angle i: j, k$ denotes the angle at vertex i of $\Delta(i, j, k)$. Two of the angles of the *convex* quadrilateral $(1, 4_3, 2, 3)$ are ϵ_3 and ϵ_4. Of the remaining two

angles, one equals or exceeds ϵ_1, and the other equals or exceeds ϵ_2 (since the three angles about a vertex of a tetrahedron satisfy the triangle inequality). It follows that $\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4 \leqq 2\pi$, and (*) holds, *a fortiori*.

Case 2. *Two largest edges of the tetrahedron are concurrent.* Assume the labelling so that these edges are (1, 4) and (3, 4), with dist (1, 4) $\geqq$ dist (3, 4). Then $\epsilon_1 = \angle 2{:}\ 3, 4$, $\epsilon_2 = \angle 3{:}\ 1, 4$, $\epsilon_3 = \angle 2{:}\ 1, 4$, while ϵ_4 may be any one of the angles of $\Delta(1, 2, 3)$, depending on which side of that triangle is largest.

The following twelve possibilities for the left-hand member of (*) must be treated:

$$(A_i) \qquad \sum_{k=1}^{4} \rho_k \epsilon_k,\ \rho_i = -1,\ \rho_k = 1,\ k \neq i,\ (i = 1, 2, 3, 4),\ \epsilon_4 = \angle 3{:}\ 1, 2,$$

and (B_i), (C_i), $(i = 1, 2, 3, 4)$, obtained from (A_i) by replacing $\angle 3{:}\ 1, 2$ by $\angle 1{:}\ 2, 3$, $\angle 2{:}\ 1, 3$, respectively.

The desired inequality is quickly obtained for every left-hand member (A_i), (B_i), (C_i), *except* C_3, by (1) use of the triangle inequality satisfied by the three face angles at any vertex of a tetrahedron, (2) replacing the sum of any two angles of a face triangle by the supplement of the remaining angle, and/or (3) replacing an angle by another (and hence smaller) angle in the same face. None of these devices is useful in the exceptional case:

$$(C_3) \qquad \angle 2{:}3, 4 + \angle 3{:}1, 4 - \angle 2{:}1, 4 + \angle 2{:}1, 3.$$

Folding $\Delta(1, 3, 4)$ about side (1, 4) into the plane of $\Delta(1, 2, 4)$ it is easily shown that $\angle 3{:}\ 1, 4 \leqq \angle 2{:}\ 1, 4$. Hence

$$\angle 2{:}3, 4 + \angle 3{:}1, 4 - \angle 2{:}1, 4 + \angle 2{:}1, 3 \leqq \angle 2{:}3, 4 + \angle 2{:}1, 4 + \angle 2{:}1, 3 - \angle 3{:}1, 4 < 2\pi,$$

and the proof is complete.

Remark. The Penning-Pauc inequality is obviously valid if e_i denotes the sum of the two largest angles of the triangle opposite vertex i $(i = 1, 2, 3, 4)$, or the sum of the largest and smallest angles of each face triangle. It is *not* valid, however, when the angle-sum is not chosen in the same way in each of the four face triangles.

2. Noncongruent tetrahedra with congruent edges. Two figures are called congruent provided there exists a one-to-one, distance-preserving correspondence between their points. Two triangles are congruent if the sides of one are congruent, respectively, to the sides of the other, *but this theorem is not valid for tetrahedra*; that is, two tetrahedra T_1, T_2 need not be congruent even though the six edges of T_1 are congruent or equal, respectively, to the six edges of T_2. In view of this fact it is pertinent to ask what is the maximum number of pairwise noncongruent tetrahedra that can be formed with six given segments for edges.

There are at most 6! tetrahedra constructible with six given edges, and each tetrahedron is congruent with 4! tetrahedra corresponding to the 4! ways of

labelling the four vertices. Hence there are at most $6!/4!=30$ tetrahedra that are pairwise noncongruent and have the same 6 edges.

DEFINITION. *Six positive numbers form a completely tetrahedral sextuple provided they are the lengths of thirty pairwise noncongruent tetrahedra.*

THEOREM 2.1. *The six positive numbers* $(a+nd)^{1/2}$, $n=0, 1, \cdots, 5$, $0<4d\leqq a$, *form a completely tetrahedral sextuple.*

Proof. Since $a\geqq 4d$, every triple of the numbers $a+nd$, $n=0, 1, \cdots, 5$, satisfies the triangle inequality, and consequently 30 pairwise noncongruent *metric* quadruples exist, the six mutual distances of the points of each quadruple being these six numbers. But it has been proved that if p_1, p_2, p_3, p_4 are a metric quadruple, then a euclidean tetrahedron exists whose edges have lengths $(p_ip_j)^{1/2}$, $(i, j=1, 2, 3, 4, i\neq j)$ ([3]; [5], pp. 130–132).

Remark 1. A metric quadruple p, q, r, s is *realizable* in euclidean three-space E_3 provided a tetrahedron exists whose vertices are congruent with them. A necessary and sufficient condition for this is that $D(p, q, r, s)\geqq 0$, where

$$D(p, q, r, s)=\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & pq^2 & pr^2 & ps^2 \\ 1 & pq^2 & 0 & qr^2 & qs^2 \\ 1 & pr^2 & qr^2 & 0 & rs^2 \\ 1 & ps^2 & qs^2 & rs^2 & 0 \end{vmatrix}.$$

Similarly, six segments are realizable in E_3 provided there exists a tetrahedron with the six segments for edges. No "elegant" necessary and sufficient condition for this (in terms of the lengths of the segments) is known nor is likely to exist, since it would involve the 30 essentially different ways that the six lengths can be distributed in the above determinant.

Remark 2. Three coplanar edges of a tetrahedron satisfy the triangle inequality, but three concurrent edges need not do so. It is easy to show, however, that *in every tetrahedron there is at least one concurrent triple of edges that satisfy the triangle inequality, and there need not be more than one such concurrent triple.*

Remark 3. There are at most $[(1/2)n(n+1)]!/(n+1)!$ pairwise noncongruent simplices in E_n with the same $(1/2)n(n+1)$ edges, $n>1$. For $n=4$, this yields 30,240, a surprising increase of the 30 tetrahedra possible when $n=3$.

Remark 4. The preceding theorem shows how a class of completely tetrahedral sextuples may be obtained. What are other ways of generating such classes? There are completely tetrahedral sextuples of integers. Do such sextuples have interesting number-theoretic properties?

Remark 5. Every triple of the six numbers 97, 98, 99, 100, 193, 194 satisfies

the triangle inequality, but *there is no tetrahedron with these six numbers for the lengths of edges.*

3. Operations on simplices. Let $\Sigma(p_0, p_1, \cdots, p_n)$, $\Sigma(q_0, q_1, \cdots, q_n)$ denote simplices of euclidean n-space E_n, with vertices p_i, q_i $(i=0, 1, \cdots, n)$, respectively. A real function $f(x, y)$ is said to be *simplicial of the first kind*, with respect to the two simplices, provided a simplex $\Sigma(r_0, r_1, \cdots, r_n)$ of E_n exists such that $r_ir_j=f(p_ip_j, q_iq_j)$, $(i, j=0, 1, \cdots, n)$.

THEOREM 3.1. *If $\Sigma(p_0, p_1, \cdots, p_n)$, $\Sigma(q_0, q_1, \cdots, q_n)$ are simplices of E_n with $p_i=q_i$, $(i=0, 1, \cdots, n)$, then $f(x, y)=[(1/2)(x+y)]^\rho$, $0\leqq\rho\leqq 1$, is simplicial of the first kind.*

A different formulation of this theorem was proved by Schoenberg [11]. It states, in effect, that if the edges of any simplex are raised to the power ρ, for ρ any number between zero and one, then another simplex exists with those numbers as lengths of edges, and such that corresponding edges of the two simplices are similarly situated.

THEOREM 3.2. *Let $\Sigma(p_0, p_1, \cdots, p_n)$ be any simplex, and let $\Sigma(q_0, q_1, \cdots, q_n)$ be an equilateral simplex with edge-length a. Then $f(x, y)=x+y$ is simplicial of the first kind with respect to those simplices.*

Proof. It suffices to prove that the quadratic form

$$Q = (1/2)\sum_{i,j=1}^{n}(r_0r_i^2 + r_0r_j^2 - r_ir_j^2)x_ix_j$$

is positive semidefinite, where $r_ir_j=p_ip_j+q_iq_j$ $(i, j=0, 1, \cdots, n)$; that is, $r_ir_j=p_ip_j+a$ $(i, j=0, 1, \cdots, n; i\neq j)$.

Substitution gives

$$Q = (1/2)\sum(p_0p_i^2 + p_0p_j^2 - p_ip_j^2)x_ix_j + a\cdot\sum(p_0p_i + p_0p_j - p_ip_j)x_ix_j + a^2\cdot(1/2)\sum x_ix_j,$$

where in each summation i and j are summed from 1 to n.

The first summand is positive semidefinite since $p_0, p_1, \cdots, p_n$ are vertices of a simplex of E_n, and the positive semidefinite character of the second summand follows from the preceding theorem applied for $\rho=1/2$. Since the third summand is obviously positive semidefinite, so is Q, and the theorem is proved.

Thus, a new simplex is formed by adding a given segment to each of the edges of a simplex.

Remark. The function $f(x, y)=x+y$ is not simplicial of the first kind with respect to two arbitrary simplices. For let p_0, p_1, p_2, p_3 be the vertices of a unit square of E_2 and $q_0=q_1$, $q_2=q_3$ the endpoints of a unit segment. It is clear that the E_3 does *not* contain four points r_0, r_1, r_2, r_3 such that $r_ir_j=p_ip_j+q_iq_j$, $(i, j=0, 1, 2, 3)$, for since $r_0r_1\cdot r_2r_3=1$, $r_0r_2\cdot r_1r_3=(1+\sqrt{2})^2$, $r_0r_3\cdot r_1r_2=4$, the four

points r_0, r_1, r_2, r_3 do not satisfy the ptolemaic inequality, which is valid for every quadruple of E_3 ([5], p. 80).

By bending the unit square a little along the diagonal joining p_0, p_2 a nondegenerate tetrahedron p_0, p_1', p_2, p_3' is obtained whose edges have the same lengths as before, except for the edge p_1', p_3', whose length $p_1'p_3'$ differs from p_1p_3 by arbitrarily little. Similarly, the unit interval $q_0=q_1$, $q_2=q_3$ may be replaced by a rectangle of arbitrarily small width, which in turn (by bending slightly along a diagonal) is replaced by a nondegenerate tetrahedron. In this way two *nondegenerate* tetrahedra are obtained for which $f(x, y)=x+y$ is not simplicial of the first kind.

It should be observed, however, that the six numbers r_ir_j, $(i, j=0, 1, 2, 3; i\neq j)$, $r_{ij}=r_{ji}$, in the example given above *are the lengths of the edges of a nondegenerate tetrahedron* $\Sigma(t_0, t_1, t_2, t_3)$ with $t_0t_1=t_2t_3=1$, $t_1t_2=t_1t_3=2$, $t_0t_2=t_0t_3=1+\sqrt{2}$. This suggests another kind of simplicial function that will be defined in this section.

THEOREM 3.3. *If* $\Sigma(p_0, p_1, \cdots, p_n)$, $\Sigma(q_0, q_1, \cdots, q_n)$ *are simplices of* E_n, *then* $f(x, y)=[x^{2\rho}+y^{2\rho}]^{1/2}$ *is a simplicial function of the first kind for every* ρ, $0\leqq\rho\leqq 1$.

Proof. Substituting $r_ir_j=[(p_ip_j)^{2\rho}+(q_iq_j)^{2\rho}]^{1/2}$ in the quadratic form Q of the preceding theorem gives

$$Q = (1/2)\sum[(p_0p_i)^{2\rho} + (p_0p_j)^{2\rho} - (p_ip_j)^{2\rho}]x_ix_j$$
$$+ (1/2)\sum[(q_0q_i)^{2\rho} + (q_0q_j)^{2\rho} - (q_iq_j)^{2\rho}]x_ix_j,$$

i and j being summed from 1 to n.

It follows from Theorem 3.1 that each of the two quadratic forms in the above equality is positive semidefinite. Hence $Q\geqq 0$ and the E_n contains a simplex $\Sigma(r_0, r_1, \cdots, r_n)$.

Remark. Let $\Sigma(p_0, p_1, \cdots, p_n)$, $\Sigma(q_0, q_1, \cdots, q_n)$ be simplices of E_n and let δ denote a given correlation between the edges of the two simplices. A function $g(x, y)$ is called *simplicial of the second kind* provided a simplex $\Sigma(r_0, r_1, \cdots, r_n)$ of E_n exists such that $r_ir_j=g(p_ip_j, \delta(p_ip_j))$, where $\delta(p_ip_j)$ denotes the length of that edge of $\Sigma(q_0, q_1, \cdots, q_n)$ that is correlated by δ with that edge of $\Sigma(p_0, p_1, \cdots, p_n)$ that joins p_i, p_j. Clearly simplicial functions of the first kind form a proper subclass of those defined in this remark, obtained by correlating those edges of the two simplices that join vertices with the same indices. Two simplices are called δ-additive provided for some correlation δ, $p_ip_j+\delta(p_ip_j)$, $(i, j=0, 1, \cdots, n; i=j)$ are the lengths of the edges of some simplex of E_n. What pairs of simplices of E_n are δ-additive?

4. An interesting class of arcs of Hilbert space. For each number ρ, $0<\rho<1$, the arc obtained from the unit segment $[0, 1]$ by defining the distance $\delta(x, y)$ of any two of its points x, y to be $xy^\rho=|x-y|^\rho$ is seen (with the help of Theorem 3.1) to be congruently contained in Hilbert space. Let $\{A_\rho\}$, $0<\rho<1$, denote

that class of Hilbert arcs.

(a) *Each subarc of each arc of class* $\{A_\rho\}$ *is nonrectifiable.*

If A_ρ^k denotes a subarc of A_ρ which corresponds (by congruence between A_ρ and the unit segment with distance $|x-y|^\rho$) to a subsegment of $[0, 1]$ whose euclidean length is k) the length $l(A)_\rho^k$ is given by

$$l(A_\rho^k) = \lim_{n\to\infty} n\cdot\epsilon(n),$$

where $\epsilon(n) = (k/n)^\rho$ is the distance in the new metric of two consecutive points in the subdivision of the subsegment into n equal (nonoverlapping) parts [1]. Hence $l(A_\rho^k) = k^\rho \cdot \lim_{n\to\infty} n^{1-\rho} = \infty$.

(b) Let x, y, z be three points of $[0, 1]$ with $x<y<z$, and put $xy=\lambda\cdot xz$, $0<\lambda<1$. Then $yz=(1-\lambda)xz$, and consequently

(i) $$\cos \angle x{:}\, y, z = [\lambda^{2\rho} - (1-\lambda)^{2\rho} + 1]/2\lambda^\rho,$$

(ii) $$\cos \angle y{:}\, x, z = [\lambda^{2\rho} + (1-\lambda)^{2\rho} - 1]/2\lambda^\rho(1-\lambda)^\rho,$$

(iii) $$\cos \angle z{:}\, x, y = [-\lambda^{2\rho} + (1-\lambda)^{2\rho} + 1]/2(1-\lambda)^\rho.$$

It is clear that none of the angles $\angle x{:}\, y, z$, $\angle y{:}\, x, z$, $\angle z{:}\, x, y$ has a limit as $y, z\to x$, except that for $\rho=1/2$, $\lim \angle y{:}\, x, z=\pi/2$. It follows that *none of the arcs* A_ρ *has either a right-hand or a left-hand tangent at any point.*

(c) The arc $A_{1/2}$ has special interest. Each three of its points are the vertices of a right triangle, and though, as seen above, it does not have a unilateral tangent at any point, *it admits a parametrization in terms of analytic functions.* The functions $f_0(t)=t$, $f_{2n-1}(t)=(\cos\ 2\pi nt)/2^{1/2}\pi n$, $f_{2n}(t)=(\sin\ 2\pi nt)/2^{1/2}\pi n$, $n=1, 2, \cdots$, are easily seen to map the unit t-interval onto an arc of Hilbert space so that if $t_1, t_2\in[0, 1]$ and p_1, p_2 are their respective corresponding points, then $p_1p_2=|t_1-t_2|^{1/2}$.

It may be conjectured that *each* arc A_ρ, $0<\rho<1$, is representable by an infinite sequence of analytic (but not elementary) functions defined on $[0, 1]$.

(d) *If* $\rho<1/2$ *each of the three angles defined in* (i), (ii), (iii) *is acute.* Clearly $\angle y{:}\, x, z$ is the largest angle, and putting $2\rho=1-\epsilon$, $0<\epsilon<1$, gives

$$\begin{aligned}\lambda^{2\rho} + (1-\lambda)^{2\rho} - 1 &= \lambda\cdot\lambda^{-\epsilon} + (1-\lambda)(1-\lambda)^{-\epsilon} - 1\\ &> \lambda + (1-\lambda) - 1 = 0.\end{aligned}$$

Similarly, it is seen that for $\rho>1/2$, every triangle of A_ρ is obtuse. *But these obtuse angles are not bounded away from* $\pi/2$. For

$$\lim_{\lambda\to 0} [\lambda^{2\rho} + (1-\lambda)^{2\rho} - 1]/2\lambda^\rho(1-\lambda)^\rho = 0.$$

Indeed, the above limit is zero for every ρ, $0<\rho<1$, and so each A_ρ contains a triangle whose maximum angle differs from $\pi/2$ by arbitrarily little.

An arc A has the Marchaud property provided a positive number δ_0 exists such that for each point p of A and each number δ, $0<\delta<\delta_0$, there are at most two points of A with distance δ from p. Marchaud proved that each euclidean arc with this property is rectifiable [8]. Clearly each of the nonrectifiable arcs A_ρ, $0<\rho<1$, has the Marchaud property, and so his theorem is not valid for arcs of Hilbert space.

5. Maximum angles of metric triples. If r, s, t are three pairwise distinct points of a metric space, let r', s', t' denote three points of the euclidean plane E_2 that correspond to them, respectively, by means of a congruence, and define the angles of the triple (r, s, t) to be the corresponding angles of the planar triple (r', s', t'). The conditions (a) $\max \angle(r, s, t)>\pi/3$, (b) $\max \angle(r, s, t)\geqq\pi/2$, (c) $\max \angle(r, s, t)\geqq\theta_0$, $\theta_0>\pi/2$, applied to triples of a (compact) metric continuum, have interesting consequences.

THEOREM 5.1. *If each three pairwise distinct points r, s, t, of a metric continuum M are such that* $\max \angle(r, s, t)>\pi/3$, *then M is an arc.*

This theorem was first conjectured by the writer in the case of locally connected metric continua, and it was proved under that restriction by Milgram [9]. Theorem 5.1 follows from results due to Choquet [6].

THEOREM 5.2. *Let C be a euclidean continuum, $p\in C$, and $\delta>0$ such that $r, s, t\in C\cdot U(p;\delta)$, $r\neq s\neq t\neq r$, imply* $\max \angle(r, s, t)\geqq\pi/2$, *where* $U(p;\delta)=[q\in C\mid pq<\delta]$. *Then C is a rectifiable arc in a neighborhood of p (that is, an open set $V(p)$, containing p, exists such that $C\cdot\overline{V}(p)$ is a rectifiable arc, where $\overline{V}(p)$ denotes the closure of $V(p)$).*

This theorem is proved in [2]. It follows from (a) and (d) of Section 4 that the restriction "euclidean" cannot be deleted from the hypothesis even if the equality sign in $\max \angle(r, s, t)\geqq\pi/2$ be dropped.

THEOREM 5.3. *Let p be a point of a metric continuum M, and suppose $\delta>0$ and $\theta_0>\pi/2$ exist such that $r, s, t\in M\cdot U(p;\delta)$, $r\neq s\neq t\neq r$, imply* $\max \angle(r, s, t)\geqq\theta_0$. *Then M is a rectifiable arc in a neighborhood of p.*

This theorem is proved in [2]. It is noteworthy that requiring the maximum angles of triples with sufficiently small diameters to be *bounded away from* $\pi/2$ (instead of demanding merely that such triples be obtuse) is all that is needed to extend the validity of Theorem 5.2 to *all* metric continua.

6. Ratio of chord to arc in metric space. The curves dealt with in classical differential geometry have the property that the limit, as two points of the curve approach a third point (along the curve), of the ratio of the length of the chord determined by the two points to the length of the intercepted arc of the curve is unity. Since examples of curves that do not possess this important feature are easily constructed, one is lead to inquire what curves do possess it at every point. It is clear that rectifiability of the curve in a neighborhood of the point is a

necessary condition, and hence only rectifiable curves enter into consideration.

Let A be a rectifiable metric arc, parameterized according to arc length; that is $A=f[0, s_0]$, where s_0 is the length $l(A)$ of A, and f maps the closed interval $[0, s_0]$ homeomorphically onto A. If $s\in[0, s_0]$, the limit under investigation, namely $\lim_{x,y\to s} f(x)f(y)/xy$, $x, y\in[0, s_0]$, is known as the *spread of f at s*. Denoting it by $f^*(s)$, it has been shown [7] that

$$(*) \qquad s_0 = l(A) = \int_0^{s_0} f^*(s)ds.$$

Since f^* is continuous wherever it exists, and $f^*(s)\leqq 1$, $s\in[0, 1]$, then (*) yields $f^*(s)\equiv 1$ in $[0, s_0]$, and the following theorem is obtained.

THEOREM 6.1. *Let A be a rectifiable metric arc. Then at each point of A the limit of the ratio of chord to arc is unity if and only if that limit exists at each point of A.*

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COMPLETELY TETRAHEDRAL SEXTUPLES

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1. Introduction. In the preceding paper (hereinafter referred to as [B]), L. M. Blumenthal defined a *completely tetrahedral sextuple* (CTS): a set of six positive numbers forms a CTS if they are the lengths of the edges of thirty pairwise noncongruent tetrahedra. (See [B], Sec. 2.) The object of this note

is to give two simple necessary and sufficient conditions that a given sextuple be a CTS. We shall also give several rather general classes of CTS's. (See [B], Sec. 2, Remark 4.)

2. Notations and preliminary remarks. We will use the symbol $\{x_0, x_1; y_0, y_1; z_0, z_1\}$ to denote the tetrahedron in which the pairs separated by semicolons are pairs of opposite edges and whose faces are formed by the triples (x_i, y_j, z_k) with $i+j+k=0$ or 2. The symmetric determinant $|r_{ij}|$ of order five in which $r_{ii}=0$, $r_{1j}=1$ for $j>1$, and $r_{23}=x_0^2$, $r_{24}=y_0^2$, $r_{25}=z_1^2$, $r_{34}=z_0^2$, $r_{35}=y_1^2$, $r_{45}=x_1^2$, will be denoted by

$$D(x_0, x_1; y_0, y_1; z_0, z_1). \tag{1}$$

If $x_0, x_1, y_0, y_1, z_0, z_1$ are six given positive numbers, any three of which satisfy the triangle inequality, then the tetrahedron $\{x_0, x_1; y_0, y_1; z_0, z_1\}$ is realizable in E_3 if and only if $D(x_0, x_1; y_0, y_1; z_0, z_1) \geqq 0$. (See [B], Sec. 2, Remark 1.)

Remark. The determinant (1) is invariant under 24 permutations of the six arguments: (a) the letters x, y, z may be permuted in any way, and (b) the indices 0 and 1 may be transposed for any two of the three letters.

Aside from the criterion just mentioned we shall also need the following one, which can be stated in terms of two-dimensional geometry. Let P_1 and P_2 be two points in E_2, which are to serve as centers of a system of bipolar coordinates in E_2. (For the sake of simpler description later on, we shall always think of P_1P_2 as being horizontal with P_1 to the left of P_2.) If Q is any point of E_2 whose distances from P_1 and P_2 are t_1 and t_2, respectively, we shall denote Q by $[t_1, t_2]_+$ or $[t_1, t_2]_-$, according as Q lies above or below the line through P_1 and P_2. (If Q lies on that line either sign may be used as subscript.) The notations $|[t_1, t_2]_+ - [u_1, u_2]_+|$ and $|[t_1, t_2]_+ - [u_1, u_2]_-|$ will be used to denote the distances between the points indicated. Now if $x_0, x_1, y_0, y_1, z_0, z_1$ are, as above, six positive numbers, satisfying the triangle inequality, then the tetrahedron $\{x_0, x_1; y_0, y_1; z_0, z_1\}$ is realizable in E_3 if and only if, in a system of bipolar coordinates with $P_1P_2=x_0$, the inequalities

$$|[y_0, z_0]_+ - [z_1, y_1]_+| \leqq x_1 \leqq |[y_0, z_0]_+ - [z_1, y_1]_-|$$

hold. This follows at once from rotating the triangle of side lengths x_0, z_1, and y_1 about its base P_1P_2.

3. A simple class of CTS's. Six positive numbers, a, b, c, d, e, f, in order to form a CTS, must necessarily be distinct from each other, and any three of them must satisfy the triangle inequality. We shall accordingly assume that

$$a > b > c > d > e > f > 0, \tag{2}$$

$$e + f \geqq a. \tag{3}$$

The two criteria of Section 2, applied to the problem of CTS's, will now be stated in the form of lemmas.

LEMMA 1. *Six numbers, satisfying conditions* (2) *and* (3), *form a CTS if and only if all thirty possible determinants* (1) *formed from them are nonnegative.*

LEMMA 2. *Six numbers, satisfying conditions* (2) *and* (3), *form a CTS if and only if, in a system of bipolar coordinates with* $P_1P_2=a$, *the inequalities*

$$|\,[w, x]_+ - [y, z]_+\,| \leqq v \leqq |\,[w, x]_+ - [y, z]_-\,| \tag{4}$$

are satisfied for all permutations v, w, x, y, z *of the letters* b, c, d, e, f.

We now state the following sufficient condition for a CTS as a consequence of Lemma 2.

THEOREM 1. *Six numbers, satisfying condition* (2) *form a CTS if* $f \geqq 2^{-1/2}a$. *In this statement the constant* $2^{-1/2}$ *is the best possible one, that is, if* $f<2^{-1/2}a$ *then* $b, c, d,$ *and* e *can be so chosen* (*in accordance with* (2)) *that* a, b, c, d, e, f *do not form a CTS.*

Assume that $f \geqq 2^{-1/2}a$. We note first that (3) is satisfied. In the bipolar coordinate system of Lemma 2, let S_+ denote the "quadrilateral" area above P_1P_2, that is bounded by arcs of the four circles with radii a and $2^{-1/2}a$ and centers P_1 and P_2, and similarly for S_- below P_1P_2. Then all points involved in the inequalities (4) lie in S_+ or in S_-. The diameter of S_+ equals $2^{-1}a<2^{-1/2}a \leqq f$, and the minimum distance between S_+ and S_- equals a. Therefore, all inequalities (4) are satisfied, and the first part of Theorem 1 follows from Lemma 2.

On the other hand, if $2^{-1}a \leqq f<2^{-1/2}a$ then $|\,[f, f]_+ - [f, f]_-\,| < a$, and it is therefore possible to choose c, d, and e near f, and b near a (all in accordance with (2)), in such a way that $|\,[c, d]_+ - [e, f]_-\,| < b$, which violates one of the inequalities (4).

Remark. The inequality $f \geqq 2^{-1/2}a$ is not a necessary condition for a CTS. On the contrary, the ratio f/a can be arbitrarily small. (See Sec. 4, Remark 5.)

4. Necessary and sufficient conditions. We first prove the following lemma.

LEMMA 3. *The minimum of the thirty determinants* (1) *that can be formed from the six quantities of inequality* (2) *is assumed by one of the following three*:

$$D_1 = D(a, b; c, d; e, f), \qquad D_2 = D(a, b; c, e; d, f), \qquad D_3 = D(a, b; c, f; d, e).$$

We write each of the thirty determinants in question in the unique standard form

$$D(a, v; w, x; y, z), \tag{5}$$

where v, w, x, y, z is a permutation of b, c, d, e, f and where $w>x$, $w>y$ and $w>z$. (See the Remark in Sec. 2 above.)

Now if $y<z$ then

(6) $D(a, v; w, x; y, z) - D(a, v; w, x; z, y) = 2(a^2 - v^2)(w^2 - x^2)(z^2 - y^2) > 0,$

and this eliminates from competition for the minimum the 15 determinants (5) in which $y<z$. We consider next the 12 of the remaining 15 determinants (5) in which $w=b$ and hence $v<b$. We use the relation

$$D(a, v; b, x; y, z) - D(a, b; v, x; y, z) = 2(a^2 - x^2)(b^2 - v^2)(a^2 + b^2 + v^2 + x^2 - 2y^2 - z^2). \tag{7}$$

By (2), the right side of (7) is positive, except possibly when $y=c$ and $z=d$ (note that $y>z$), and in this case we use the relation

$$D(a, v; b, x; c, d) - D(a, d; b, x; c, v) = 2(a^2 - c^2)(d^2 - v^2)(a^2 - b^2 + c^2 + d^2 + v^2 - 2x^2), \tag{8}$$

whose right-hand member is positive. Thus (7) and (8) eliminate the 12 above-mentioned determinants (5) from competition for the minimum. The three remaining determinants are those in which $v=b$ (and $y>z$), and these are the three determinants D_1, D_2, and D_3.

Remark 1. Which one of the three determinants of Lemma 3 actually assumes the minimum can be seen from applying (7) and (8) to the differences D_2-D_1 and D_3-D_2, respectively: the answer is D_1, D_2, or D_3, according as $c^2+d^2+e^2+f^2$ is greater than $2a^2+b^2$, between a^2+2b^2 and $2a^2+b^2$, or less than a^2+2b^2, respectively.

THEOREM 2. *Six numbers, satisfying conditions* (2) *and* (3), *form a CTS if and only if, in a system of bipolar coordinates with* $P_1P_2=a$, *the inequality*

$$|\,[c, d]_+ - [e, f]_-\,| \geqq b \tag{9}$$

holds.

By Lemma 2, it suffices to prove the sufficiency of (9). Let T be the "triangular" area which is bounded below by part of the segment P_1P_2 and on the right and left by circular arcs of radius c with centers at P_1 and P_2, respectively. The diameter of T is less than $c<b$. Therefore, if w, x, y, z is any permutation of c, d, e, f, we have, by (2),

$$|\,[w, x]_+ - [y, z]_+\,| < b, \tag{10}$$

since all points occurring in (10) lie in T.

Application of (10) to $[c, d]_+$ and $[e, f]_+$ together with (9) gives $D_3 \geqq 0$. For obvious geometrical reasons, (9) and (2) imply that *a fortiori* $|\,[c, d]_+ - [f, e]_-\,| > b$; and this combined with (10), applied to $[c, d]_+$ and $[f, e]_+$, gives $D_2>0$. Finally, by (6) and (2), $D(a, b; c, f; e, d) > D_3 \geqq 0$, hence $|\,[c, e]_+ - [d, f]_-\,| > b$ and *a fortiori* $|\,[c, e]_+ - [f, d]_-\,| > b$; the last inequality, together with (10), applied to $[c, e]_+$ and $[f, d]_+$, gives $D_1>0$. Theorem 2 now follows from Lemmas 3 and 1.

THEOREM 3. *Six numbers, satisfying conditions* (2) *and* (3), *form a CTS if and only if* $D_3 = D(a, b; c, f; d, e) \geqq 0$, *that is, if and only if the tetrahedron in which the faces are formed by the triples* (a, c, d), (a, e, f), (b, c, e) *and* (b, d, f) *is realizable in* E_3.

The proof is immediate, since the fact that $D_3 \geqq 0$ implies that, in a system of bipolar coordinates with $P_1P_2 = a$, the inequality $|\,[c, d]_+ - [e, f]_-| \geqq b$ holds and, consequently, Theorem 2 becomes applicable.

Remark 2. We note without proof that in Theorem 3 condition (3) is not needed as part of the hypothesis; in the particular case of the determinant D_3, it can be shown that (3) follows from $D_3 \geqq 0$ and (2).

Remark 3. The determinant D_3 is the only one of the thirty determinants (5) which can serve the purpose of Theorem 3. This can be seen from the example of the sextuple 120, 110, 100, 90, 80, 61, for which $D_3 < 0$, while the other 29 determinants are positive.

The following remarks are direct applications of Theorem 3.

Remark 4. Six positive numbers $x+5$, $x+4$, $x+3$, $x+2$, $x+1$, x form a CTS if and only if x is greater than or equal to the positive root of the equation $D(x+5, x+4; x+3, x; x+2, x+1) = 0$; this root lies between 6.09 and 6.10. In particular, six successive integers form a CTS if and only if the largest of them is at least 12. (Generalization to any arithmetic progression is obvious.) The CTS 12, 11, 10, 9, 8, 7 is moreover the smallest integral CTS; there is no other integral CTS whose largest member is less than or equal to 12.

Remark 5. There are CTS's in which the ratio f/a (see (2)) is arbitrarily small. This follows from the relation $D(1, 1; 1, \epsilon; 1, 1) = \epsilon^2(3-\epsilon^2) > 0$ for any ϵ, $0 < \epsilon < 1$. Choose $\delta = \delta(\epsilon) > 0$ so that $1-4\delta > \epsilon$ and $D(1, 1-\delta; 1-2\delta, \epsilon; 1-3\delta, 1-4\delta) > 0$.

Remark 6. An investigation of the case discussed in [B], Theorem 2.1, leads to the result that the sextuple $(a+nd)^{1/2}$, $n = 0, 1, 2, 3, 4, 5$, with $a > 0$, $d > 0$, is a CTS if and only if a/d is greater than or equal to a cubic irrationality, which lies between 1.91 and 1.92.

DANDELIN, LOBAČEVSKIĬ, OR GRAEFFE?

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In the Russian translation of reference [5], the designation of Graeffe's method is changed to "Lobačevskiĭ's method," though not without a conscientious footnote calling attention to the fact. Since some texts refer to the "Dandelin-Graeffe method," the question who did what and when seemed of some interest.

The "when" question is readily answered. The three authors published in 1826 [2], 1834 [6], and 1837 [4], respectively. Considering this, the natural conclusion would be that Dandelin has clear priority, and that the attachment of Graeffe's name to the method must have been one of those unfortunate historical accidents that do occur at times. But if justice is to be attempted, why pass over Dandelin for Lobačevskiĭ? An interesting, and curious, paper by Rogačenko [8] discusses the question at some length, not only with analyses of the papers by the three authors, but in quoting from writers who uncritically assign the credit to Graeffe, as well as others who have other opinions.

Rogačenko freely admits that Dandelin published in 1826 a paper in which root squaring is proposed. He seems to consider it important that although Lobačevskiĭ's *Algebra* [6] bears the date 1834, it was actually in the hands of the censor in 1832. But he builds his case upon the assertion that Dandelin's paper was concerned primarily with Newton's method, and that root squaring is introduced only as a device for speeding the convergence of the Newton iteration. It is quite true that the main part of the paper does discuss Newton's method, and it is recommended that it be accompanied by *regula falsi* so that one has always an upper and a lower bound for the root. Dandelin then considers the possibility of accelerating both processes by applying them to the equation whose roots are the squares of those of the original. The method he proposes for doing this is to form the product $f(x)f(-x)$, where $f(x)=0$ is the original equation (note that this is not quite the way one does it now). He mentions also another device for accelerating convergence, by use of osculating parabolas instead of tangents and chords. Finally, in the paper proper, he comments that one could equally well take 4th powers, 8th powers, or whatever, all of which, however, do subordinate the algebraic root squaring to the geometric tangents and chords. However, Dandelin had afterthoughts, which he recorded in four appendices, and in the second of these he goes further into the matter of root squaring, making two important observations that are not always to be found in modern treatments.

The first of these observations is also reported by Rogačenko, but the second is not. The first is that if the zeros of a polynomial are widely separated into one group of very large modulus, and one of very small modulus, then the equation which remains when final terms are dropped is approximately satisfied by the large zeros. In particular, if there is only one large zero, keep the first two terms; if there are two, keep three. The second observation is that if one considers the power 2^p sufficiently high, then form the power $1+2^p$, and one gets the zeros themselves as quotients without root extraction. It is hard to reconcile the presence of these considerations with the contention that Dandelin considered root squaring an auxiliary device only, and not a possible method in itself. It is true that our critic does level another charge against Dandelin, but the relevance is obscure. It is that Dandelin did not base his conclusions upon properties of symmetric polynomials.

In the editorial introduction to Volume 4 of Lobačevskiĭ's works, Čebotarev

[1] opines that Lobačevskiĭ deserves priority over Graeffe, but fails to make an argument on Dandelin. He tells us he does not have available the 1837 monograph of Graeffe [4] but he inserts a translation of an earlier paper [3] which does deal with symmetric functions, and proves a convergence theorem, but does not describe the method. And Lobačevskiĭ does, indeed, on the final page of this edition of his *Algebra* which had just appeared in 1834, make the following suggestion: Consider equations

$$x^n - a_1x^{n-1} + \cdots = 0, \qquad y^n - A_1y^{n-1} + \cdots = 0,$$

where $A_1 = a_1^2 - 2a_2$, $A_2 = a_2^2 - 2a_1a_3 + 2a_4$, $\cdots$. Then the roots of the second equation are the squares of the roots of the first. In justification of the statement he refers to the identities he had previously established relating sums of powers with the coefficients. Notably, he did not refer this to the also developed properties of the product

$$f(x)f(\omega x) \cdots,$$

where ω is a root of unity. Finally, in illustration he finds the largest root of a quintic.

Why only the largest root, and what about the others? On this there is no clue, except, possibly a negative one, which is this, that in the final equation he computes only A_1, and in the intermediate equations only those coefficients that affect A_1. If he thought of this as a method for obtaining *all* roots *simultaneously*, he gives no hint of the fact. But at least he uses symmetric functions.

And now, where does Graeffe fit in, and why did anyone ever mention him? He fits in for having thought of separating the even and odd powers. A simple notion, but effective, and it is just what everyone does today. And so, Ostrowsky, in his elaborate treatment of the method [7], attaches the name of Graeffe, and begins by summarizing the contributions of these three, along with those of Encke and several others who further investigated and elaborated the technique. Whether this is justice the reader must decide for himself.

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A GENERATING FUNCTION FOR $\sigma_k(n)$

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1. Introduction. In this paper we shall be concerned with certain arithmetical properties of formal infinite series. By formal infinite series, we mean that the convergence of the infinite series under discussion is irrelevant. For a more detailed discussion of this point of view, see [2], pages 251–253 or [4], Chapter 1.

It is well known ([1], pp. 279–280) that

$$(1.1) \qquad \sum_{n=1}^{\infty} \frac{nx^n}{1-x^n} = \sum_{n=1}^{\infty} \sigma(n)x^n, \qquad \sum_{n=1}^{\infty} \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} \tau(n)x^n,$$

where $\sigma(n)$ is the sum of the divisors of n and $\tau(n)$ is the number of divisors of n.

In addition, Harris has published [3] the identity:

$$(1.2) \qquad \sum_{n=1}^{\infty} \frac{5-(-1)^n}{2} \frac{nx^{n-1}}{1-x^n} = \sum_{n=1}^{\infty} \sigma(2n)x^{n-1}.$$

The purpose of this paper is to generalize the identities (1.1) and (1.2). Our generalization proceeds as follows: We seek an arithmetic function, $f_k(an+b)$, with $a>0$ and $b\geqq 0$ so that

$$(1.3) \qquad \sum_{n=1}^{\infty} \frac{f_k(an+b)x^{an+b}}{1-x^{an+b}} = \sum_{n=1}^{\infty} \sigma_k(an+b)x^{an+b},$$

where $\sigma_k(n)$ is the sum of the kth powers of the divisors of n. We wish to find the conditions under which $f_k(an+b)$ exists, and when it does, to characterize it.

In Section 2 are various lemmata which help us attain our goal. A principal result is that if there is an arithmetic function $f_k(an+b)$ satisfying (1.3) with $b>0$, then $a=(a, b)$.

Section 3 leads us to a characterization of $f_k(an+b)$, when it exists, together with certain of its properties. Equating corresponding coefficients on both sides of (1.3) yields the novel identity of Theorem 4.

In Section 4 are presented several examples. In particular a practical system for computing a table of the sum of the kth powers of the divisors of n is devised. The method used is similar to the use of the sieve of Eratosthenes in computing number-divisor tables. The main feature of the system is that it carries its own check with it.

2. Some lemmata. The first lemma is a result of Cesàro ([1], p. 127).

LEMMA 1. *If $f(n)$ is an arithmetic function, then*

$$\sum_{n=1}^{\infty} \frac{f(n)x^n}{1-x^n} = \sum_{n=1}^{\infty} F(n)x^n,$$

where $F(n)=\sum_{d|n} f(d)$.

The next lemma is a slight modification of a well-known result. A proof can be found in [2], page 235.

LEMMA 2. *If a divides m and $\mu(m)$ is the Möbius function, then*

$$\sum_{d|(m/a)} \mu(d) = \begin{cases} 1 & \text{if } m = a, \\ 0 & \text{if } m > a. \end{cases}$$

Lemma 3 requires the introduction of the following definition: If $\mu(n)$ is the Möbius function, we define

$$\bar{\mu}(m/a) = \begin{cases} \mu(m/a) & \text{if } a \mid m, \\ 0 & \text{if } a \nmid m. \end{cases}$$

LEMMA 3. *With $\bar{\mu}(m/a)$ defined as above, we have*

$$\sum_{m=1}^{\infty} \frac{\bar{\mu}(m/a)x^m}{1-x^m} = x^a.$$

Proof. By Lemma 1,

$$\sum_{m=1}^{\infty} \frac{\bar{\mu}(m/a)x^m}{1-x^m} = \sum_{m=1}^{\infty} \left(\sum_{d|m} \bar{\mu}(d/a) \right) x^m = \sum_{\substack{m=1 \\ a|m}}^{\infty} \left(\sum_{d|(m/a)} \mu(d) \right) x^m.$$

Then by Lemma 2,

$$\sum_{\substack{m=1 \\ a|m}}^{\infty} \left(\sum_{d|(m/a)} \mu(d) \right) x^m = x^a.$$

LEMMA 4. *If there is an arithmetic function $f_k(an+b)$ with $b>0$ such that* (1.3) *is satisfied, then $a=(a, b)$.*

Proof. Set $g=(a, b)$ so that $1=(a/g, b/g)$. There is an N_0 so that $(a/g)N_0 + (b/g) = p$, where p is a prime. Certainly x^{gp} occurs on the right side of (1.3) with nonzero coefficient. Expanding the left side of (1.3), we see that the coefficient of x^{aN+b} is $\sum f_k(an+b)$ summed over all n for which $(an+b) \mid (aN+b)$. Thus the coefficient of x^{gp} in the expansion of the left side of (1.3) is just $f_k(aN_0+b)$ and hence is not zero.

Now consider the coefficient of x^{2gp} in the expansion of the left side of (1.3). This coefficient is $\sum f_k(an+b)$ summed over all n for which $(an+b) \mid (2gp)$. This includes $n=N_0$ but may include other values of n. If $n \neq N_0$ and $(an+b) \mid (2gp)$, then a short and obvious argument leads us to $a=g$. On the other hand, if N_0 is the only n for which $(an+b) \mid (2gp)$, we have the coefficient of x^{2gp} as just $f_k(aN_0+b)$, which we have just shown is not zero. Hence x^{2gp} must occur on the right side. This gives us an n_1 such that $an_1+b=2gp$. Combining $an_1+b=2gp$ and $aN_0+b=gp$ yields $(a/g)(n_1-N_0)=p$. Thus a/g divides p, which implies $a=g$. In all cases, then, $a=g$.

3. Theorems. The first theorem reveals that a unique $f_k(an+b)$ satisfying (1.3) exists if $b=0$.

THEOREM 1. *If $f_k(a, n)$ is an arithmetic function for which*

$$\sum_{n=1}^{\infty} \frac{f_k(a, n)x^n}{1-x^n} = \sum_{n=1}^{\infty} \sigma_k(an)x^n,$$

then

$$f_k(a, n) = s^k\sigma_k(r)n^k, \tag{3.1}$$

where r is the largest factor of a for which $(r, n)=1$ and $a=rs$, and conversely.

Proof. By Lemma 1,

$$\sum_{d|n} f_k(a, d) = \sigma_k(an).$$

By the Möbius inversion formula,

$$\begin{aligned} f_k(a, n) &= \sum_{d|n} \sigma_k(an/d)\mu(d) \\ &= \sum_{d|sn} \sigma_k(an/d)\mu(d) \end{aligned} \tag{3.2}$$

because $\mu(d)=0$ if d is not square free. Continuing we can write

$$f_k(a, n) = \sigma_k(r) \sum_{d|sn} \sigma_k(sn/d)\mu(d) \tag{3.3}$$

since r and sn/d are relatively prime.

Now consider

$$\sum_{d|m} d^k = \sigma_k(m).$$

From the Möbius inversion formula

$$m^k = \sum_{d|m} \sigma_k(m/d)\mu(d). \tag{3.4}$$

Substituting from (3.4) into (3.3) gives us $f_k(a, n)=\sigma_k(r)(sn)^k$ as was to be proved. Retracing steps yields the converse.

THEOREM 2. *Let us now set*

$$g_k(t; a, m) = f_k(a, m) - \sum_{\alpha=1}^{t} \sigma_k(\alpha a)\bar{\mu}(m/\alpha)$$

with the usual restriction that the sum is zero if $t=0$. Then

$$\sum_{m=1}^{\infty} \frac{g_k(t; a, m)x^m}{1-x^m} = \sum_{m=t+1}^{\infty} \sigma_k(am)x^m.$$

Proof. By Theorem 1

$$\sum_{m=t+1}^{\infty} \sigma_k(am)x^m = \sum_{m=1}^{\infty} \sigma_k(am)x^m - \sum_{\alpha=1}^{t} \sigma_k(a\alpha)x^\alpha$$

$$= \sum_{m=1}^{\infty} \frac{f_k(a, m)x^m}{1 - x^m} - \sum_{\alpha=1}^{t} \sigma_k(a\alpha)x^\alpha.$$

With the aid of Lemma 3,

$$\sum_{m=t+1}^{\infty} \sigma_k(am)x^m = \sum_{m=1}^{\infty} \frac{f_k(a, m)x^m}{1 - x^m} - \sum_{m=1}^{\infty} \frac{\sum_{\alpha=1}^{t} \sigma_k(a\alpha)\mu(m/\alpha)x^m}{1 - x^m}$$

$$= \sum_{m=1}^{\infty} \frac{g_k(t; a, m)x^m}{1 - x^m}.$$

COROLLARY. *If $m \leqq t$, then $g_k(t; a, m) = 0$. Hence*

$$\sum_{m=t+1}^{\infty} \frac{g_k(t; a, m)x^m}{1 - x^m} = \sum_{m=t+1}^{\infty} \sigma_k(am)x^m.$$

Proof. If $m \leqq t$, then

$$\sum_{\alpha=1}^{t} \sigma_k(\alpha a)\mu(m/\alpha) = \sum_{d|m} \sigma_k(ad)\mu(m/d) = \sum_{d|m} \sigma_k(am/d)\mu(d) = f_k(a, m) \text{ by (3.2).}$$

THEOREM 3. *If $b = at$, set $h_k(t; a, n) = g_k(t; a, n+t)$. Then*

$$\sum_{n=1}^{\infty} \frac{h_k(t; a, n)x^{an+b}}{1 - x^{an+b}} = \sum_{n=1}^{\infty} \sigma_k(an + b)x^{an+b}.$$

Proof. Theorem 3 is merely Theorem 2 reworded. It is set up as a separate theorem because it exhibits an arithmetic function, $h_k(t; a, n) = f_k(an+b)$, which satisfies (1.3) if $a = (a, b)$. In view of Lemma 4, we now have proved the existence of $f_k(an+b)$ if and only if $a = (a, b)$.

THEOREM 4. *If $b = at$, set*

$$\delta_k(a, m) = \begin{cases} 0 & m \leqq t, \\ \sigma_k(am) & m > t. \end{cases}$$

Then

$$\sum_{d|m} g_k(t; a, d) = \delta_k(a, m).$$

Proof. By Theorem 2 and the definition of $\delta_k(a, m)$, we can write

$$\sum_{m=1}^{\infty} \frac{g_k(t; a, m)x^m}{1 - x^m} = \sum_{m=1}^{\infty} \delta_k(a, m)x^m.$$

Then by Lemma 1

$$\delta_k(a, m) = \sum_{d|m} g_k(t; a, d).$$

THEOREM 5. *Set $\phi_k(a, n) = f_k(a, n)/n^k$, where $f_k(a, n)$ is defined by* (3.1). *If P is the product of the distinct prime factors of a (including* 1), *then $\phi_k(a, n)$ is periodic in n with least period P.*

Proof. First of all it is easy to establish that if g is any factor of a such that $(g, n) = 1$, then $(g, n + P) = 1$ and conversely. Thus (3.1) shows $\phi_k(a, n) = \phi_k(a, n+P)$ for all n, and therefore $\phi_k(a, n)$ is periodic with least period $R \leqq P$.

If $\phi_k(a, n)$ has least period R, then $\phi_k(a, n) = \phi_k(a, n+R)$ for all n. In particular take $n = a$ to obtain

$$a^k\sigma_k(1) = s^k\sigma_k(r),$$

where r is the largest factor of a such that $(r, a+R) = 1$ and $a = rs$. Thus we have $\sigma_k(r) = (a/s)^k = r^k$ which implies that $r = 1$. Therefore unity is the largest factor of a which is relatively prime to $a+R$. It follows that every prime factor of a divides R. Hence $R \geqq P$ whereupon $R = P$ as asserted.

4. Examples. The first example illustrates Theorem 1. Let us take $k = 2$ and $a = 12$. By Theorem 5, $f_2(12, n)/n^2$ is periodic with least period 6. Hence if we set

$$\gamma_n = 210,\ 160,\ 189,\ 160,\ 210,\ 144$$

when

$$n \equiv 1,\quad 2,\quad 3,\quad 4,\quad 5,\quad 0,\ \pmod 6,$$

respectively, then

$$\sum_{n=1}^{\infty} \frac{\gamma_n n^2 x^n}{1 - x^n} = \sum_{n=1}^{\infty} \sigma_2(12n)x^n.$$

The second example is on Theorem 4. Take $a = 15$, $k = 1$, $t = 3$, and $m = 12$. By Theorem 4

$$\sum_{d|12} g_1(3; 15, d) = \sigma(180).$$

By direct computation

$$\sum_{d|12} g_1(3; 15, d) = \sum_{\substack{d|12 \\ d \geq 4}} g_1(3; 15, d)$$

$$= 4\sigma(15) + \sigma(30) + 18\sigma(5) - \sigma(15) + \sigma(30) + \sigma(45) + 36\sigma(5) - \sigma(30)$$

$$= 546 = \sigma(180).$$

For the last example we shall describe how to construct a table for $\sigma_k(n)$. By Theorem 1

$$\sum_{n=1}^{\infty} \frac{f_k(a, n)x^n}{1 - x^n} = \sum_{n=1}^{\infty} \sigma_k(an)x^n.$$

Equating corresponding coefficients yields the following systematic scheme:

x	x^2	x^3	x^4	$\cdots$
$f_k(a, 1)$	$f_k(a, 1)$ $f_k(a, 2)$	$f_k(a, 1)$ $f_k(a, 3)$	$f_k(a, 1)$ $f_k(a, 2)$ $f_k(a, 4)$	
$\sigma_k(a)$	$\sigma_k(2a)$	$\sigma_k(3a)$	$\sigma_k(4a)$	$\cdots$

If a is the prime p, then by the definition of $f_k(p, n)$ in (3.1), and because $f_k(p, n)/n^k$ is periodic with least period p by Theorem 5, we have

$$f_k(p, n) = \begin{cases} p^k n^k \text{ if } p \text{ divides } n, \\ (p^k + 1)n^k \text{ otherwise.} \end{cases}$$

The considerations of the last two paragraphs are illustrated by computing $\sigma_2(an)$ for $a = 5$ and $n = 1, 2, 3, 4, 5$, and 6.

an	5	10	15	20	25	30
	26	26 104	26 234	26 104 416	26 625	26 104 234 936
$\sigma_2(an)$	26	130	260	546	651	1300

If for each prime p we compute $\sigma_k(p)$, $\sigma_k(2p)$, $\sigma_k(3p)$, $\cdots$, then $\sigma_k(n)$ will be computed at least twice unless n is unity or a prime power, but these special cases are easily checked directly.

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ON FORMING PARTIAL DIFFERENTIAL EQUATIONS*

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1. Introduction. Although there are many texts dealing with the formation of partial differential equations (p.d.e.'s) by the elimination of arbitrary functions, there is no systematic theory on the subject for equations of order greater than 1, and no conditions by which it can be guaranteed that a given equation involving arbitrary functions will have a p.d.e. of lowest order or of a specified order, *e.g.*, the same order as the number of arbitrary functions.

Since a p.d.e. for most functions can always be found by a sufficient number of differentiations it will be the purpose of this paper to try to determine the p.d.e. of lowest order in each case.

The letters p, q, r, s, and t will have their usual connotations and all work will be done in three dimensions.

The first part of this paper deals with explicit functions and the second part with implicit functions.

2. Explicit functions. Consider

$$z = \sum_{i=1}^{n} u_i(x, y)F_i[v(x, y)], \tag{1}$$

where the $u_i(x, y)$ and $v(x, y)$ are given functions of class C^n in both x and y and the F_i are arbitrary functions of class C', for $i=1, \cdots, n$.

From (1) we obtain

$$v_y p - v_x q = \sum_{i=1}^{n} (u_{ix}v_y - u_{iy}v_x)F_i,$$

and setting $z_2 = u_1(v_y p - v_x q) - z(u_{1x}v_y - u_{1y}v_x)$, $\phi_i = u_i(u_{ix}v_y - u_{iy}v_x) - u_i(v_y p - v_x g)$, $i=2, \cdots, n$, $z_2 = \sum_{i=2}^{n} \phi_i F_i$, which has the same form as (1) but contains one less arbitrary function. Proceeding in the same manner, we can eliminate the remaining $n-1$ arbitrary functions yielding a p.d.e. for (1) of at most nth order.

Functions of the form

$$z = \prod_{i=1}^{n} \{G_i[v(x, y)]\}^{u_i(x,y)} \tag{2}$$

are special cases of (1); for (2) may be written $\log z = \sum_{i=1}^{n} u_i \log G_i$, which takes the form of (1) upon setting $z_1 = \log z$ and $F_i = \log G_i$, $i=1, \cdots, n$.

In general, we can find a p.d.e. for functions of the type

$$z = \sum_{i=1}^{n} u_i(x, y)F_i[v_i(x, y)] \tag{3}$$

* This problem was suggested to the author by Professor H. J. Zimmerberg of Rutgers University.

by taking a sufficient number of partial derivatives. The totality of equations obtained by taking partial derivatives of (3) to the jth order is $(j+1)(j+2)/2$ for $j=1, \cdots, n$. Since each order of differentiation of (3) introduces n new arbitrary functions, there are at most $n(j+1)$ arbitrary functions to be eliminated from the $(j+1)(j+2)/2$ equations. Since $(j+1)(j+2)>2n(j+1)$ for $j>2n-2$, the p.d.e. of (3) will be at most of order $2n-1$. However we shall investigate the functions of the form (3) for which it is possible to obtain a p.d.e. of at most order n.

For $n=1$ it is always possible, and the p.d.e. for (3) is $u(v_y p - v_x q) - (u_x v_y - u_y v_x)z = 0$.

Consider $n=2$. After obtaining the partials of (3) up to and including the second order we have (with equation (3)) six equations involving linearly F_i, F_i', F_i'', $i=1, 2$. A necessary and sufficient condition that these equations be functionally dependent is that the 6×6 determinant of the F_i and their derivatives be identically zero. Let D denote this determinant.

After some lengthy but straightforward calculations*,

$$\begin{aligned}\frac{D}{(u_1u_2)^3} = J\Bigg[J\Bigg\{v_{1y}v_{2y}\left[\left(\frac{u_{2x}}{u_2}\right)^2 - \frac{u_{2xx}}{u_2} - \left(\frac{u_{1x}}{u_1}\right)^2 + \frac{u_{1xx}}{u_1}\right] \\ + (v_{1x}v_{2y} + v_{1y}v_{2x})\left[\frac{u_{1x}u_{1y}}{u_1^2} - \frac{u_{1xy}}{u_1} - \frac{u_{2x}u_{2y}}{u_2^2} + \frac{u_{2xy}}{u_2}\right] \\ + v_{1x}v_{2x}\left[\left(\frac{u_{2y}}{u_2}\right)^2 - \frac{u_{2yy}}{u_2} - \left(\frac{u_{1y}}{u_1}\right)^2 + \frac{u_{1yy}}{u_1}\right]\Bigg\} \\ + P(1, 2, y)\left(\frac{u_{2x}}{u_2} - \frac{u_{1x}}{u_1}\right) - P(1, 2, x)\left(\frac{u_{2y}}{u_2} - \frac{u_{1y}}{u_1}\right)\Bigg],\end{aligned}$$

where

$$\begin{aligned}J &= v_{1y}v_{2x} - v_{1x}v_{2y}, \\ P(i, j, a) &= v_{1y}v_{2y}(v_{ia}v_{jxx} - v_{ja}v_{ixx}) + v_{1x}v_{2x}(v_{ia}v_{jyy} - v_{ja}v_{iyy}) \\ &\quad + (v_{1y}v_{2x} + v_{1x}v_{2y})(v_{ja}v_{ixy} - v_{ia}v_{jxy}).\end{aligned}$$

Observe that $\partial/\partial a(u_{ia}/u_i) = (u_i u_{iaa} - u_{ia}^2)/u_i^2$, $\partial/\partial y(u_{ix}/u_i) = \partial/\partial x(u_{iy}/u_i) = (u_i u_{ixy} - u_{ix}u_{iy})/u_i^2$, $u_{ia}/u_i = \partial/\partial a(\log u_i)$, $a=x, y$; $i=1, 2$; substitution of these relations into the above equation gives, after some simplification

$$(4) \qquad D/(u_1u_2)^3 = J\{RD_x^2 w + SD_xD_y w + TD_y^2 w + PD_x w + QD_y w\} \equiv 0,$$

where $w = \log(u_2/u_1)$, $R = -v_{1y}v_{2y}J$, $S = (v_{1x}v_{2y} + v_{1y}v_{2x})J$, $T = -v_{1x}v_{2x}J$, $P = P(1, 2, y)$, and $Q = -P(1, 2, x)$.

If $J\equiv 0$, then F_1 and F_2 are functionally dependent and a second-order p.d.e. exists. Assume $J\not\equiv 0$. Then J is at most conditionally zero which, from the con-

* It is not recommended that the reader attempt to perform the calculations.

tinuity of J, implies that there is a region where $J \neq 0$.

The part of (4) in braces is a second-order linear p.d.e. which can be solved by means of Laplace's transformation (*c.f.* [7], p. 182), using $x=x(u, v)$, $y=y(u, v)$. Using the notations of [7], and substituting in $R\theta^2+S\theta+T=0$ we obtain $\theta=v_{1x}/v_{1y}$, v_{2x}/v_{2y}. It is apparent that the simple integrals $u=v_1$, $v=v_2$ will satisfy equations (7) and (8) of [7].

Substituting in the system (5) of [7] we obtain $R'=T'=P'=Q'=0$ and $S'=J^2$. (4) of [7] then becomes $S's'=0$, where $s'=\partial^2 w/\partial u \partial v$.

Therefore we have as a solution,

$$u_2(x, y) = u_1(x, y) \exp \{f_1[v_1(x, y)] + f_2[v_2(x, y)]\}, \tag{5}$$

where the f_i are any functions of class C^2.

We have therefore proved the following

THEOREM 1. *A necessary and sufficient condition that a partial differential equation of at most second order exists for* (3) *with* $n=2$, *where the* $u_i \neq 0$, $v_i \neq 0$ *are given functions of class* C^2 *in* x *and* y, *and the* F_i *are arbitrary functions of class* C^2 $(i=1, 2)$ *in some region* S, *is that either* $v_{1y}v_{2x}-v_{1x}v_{2y} \equiv 0$, *in which case the* F_i *are functionally dependent, or* (5) *holds.*

The general solutions of the second order reducible homogeneous p.d.e 's $(\beta_1 D_x-\alpha_1 D_y-\gamma_1)(\beta_2 D_x-\alpha_2 D_y-\gamma_2)z=0$, for all possible values of the constants α_i, β_i, γ_i can readily be shown to form a proper subset of the general family (5) obtained by setting $v_i=a_i x+b_i y$, $i=1, 2$.

Consider (3) for $n>2$. Taking all of the partial derivatives of (3) up to and including the nth order gives $(n+1)(n+2)/2$ equations involving linearly $n(n+1)$ arbitrary functions. Consider the $[(n+1)(n+2)/2] \times n(n+1)$ matrix M of the coefficients of these $n(n+1)$ functions. Since $(n+1)(n+2)/2 < n(n+1)$ for $n>2$, we have: A necessary and sufficient condition that the p.d.e. of (3) is at most of order n is that the rank of M be less than $(n+1)(n+2)$.

3. Implicit functions. For

$$F(u_1, \cdots, u_n) = 0, \tag{6}$$

$u_i=u_i(x_1, \cdots, x_n, z)$, F arbitrary, u_i given, Ince ([4], p. 9) has shown that the p.d.e. is first order and exhibits it. For $n=2$, $x_1=x$, $x_2=y$, [2] p. 227, [6] p. 212, [7] p. 64, and [11] p. 294 show that the p.d.e. is of first order and exhibit it.

Consider $\phi(x, y, z)=\sum_{i=1}^{n} u_i(x, y, z)F_i[v_i(x, y, z)]$, where ϕ, u_i, v_i are given and F_i arbitrary suitably differentiable functions. For $n=1$, a first order p.d.e. clearly exists. For $n=2$, a result analogous to Theorem 1 holds, with $u_i(x, y)$, $v_i(x, y)$ replaced by $u_i(x, y, z)$, $v_i(x, y, z)$ in (5). For $n>2$ the discussion at the end of Section 2 is true also for the implicit functions.

4. Conclusion. The partial listing of texts on p.d.e.'s in the bibliography except [7] contain only problems which are all special cases of (3) for $n=1$, Theorem 1, or (6) for $n=2$. The exceptions in [7] are problems 19 and 20 on

page 65, and it can be readily verified that they satisfy the condition stated at the end of Section 2.

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NOTE ON THE METHOD OF CONTRACTANTS

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An apparently new and interesting method for the evaluation of determinants was recently described by Macmillan in [1]. It was stated there that this method (i) is very easy to learn and use, (ii) involves as few arithmetical operations as the most rapid method of pivotal condensation, (iii) can be programmed easily for a digital computer. We have applied this method to a number of cases, and have found it in general to be as attractive as indicated above.

The technique is easily described as follows: Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{1}$$

be a matrix whose determinant $\det A$ is to be determined. This may be accomplished by the calculation of a sequence of matrices of successively lower order: First

$$A_1 = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1,n-1}^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} & \cdots & a_{2,n-1}^{(1)} \\ \cdot & \cdot & & \cdot \\ a_{n-1,1}^{(1)} & a_{n-1,2}^{(1)} & \cdots & a_{n-1,n-1}^{(1)} \end{bmatrix} \tag{2}$$

is obtained from A by means of all the second order minors:

$$a_{ij}^{(1)} = [a_{ij}a_{i+1,j+1} - a_{i,j+1}a_{i+1,j}] \tag{3}$$

$i, j=1, \cdots, n-1$. Next A_1 is contracted into A_2, where

$$A_2 = \begin{bmatrix} a_{11}^{(2)} & a_{12}^{(2)} & \cdots & a_{1,n-2}^{(2)} \\ a_{21}^{(2)} & a_{22}^{(2)} & \cdots & a_{2,n-2}^{(2)} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{n-2,1}^{(2)} & a_{n-2,2}^{(2)} & \cdots & a_{n-2,n-2}^{(2)} \end{bmatrix} \tag{4}$$

and

$$a_{ij}^{(2)} = \frac{1}{a_{i+1,j+1}} [a_{ij}^{(1)} a_{i+1,j+1}^{(1)} - a_{i,j+1}^{(1)} a_{i+1,j}^{(1)}] \tag{5}$$

$i, j=1, \cdots, n-2$.

Thus, in general, $A_{k+1} \equiv (a_{ij}^{(k+1)})$ is obtained from the $A_k \equiv (a_{ij}^{(k)})$, $i, j=1, \cdots, n-k$; $k=0, 1, \cdots, n-1$, by the simple algorithm

$$a_{ij}^{(k+1)} = \frac{1}{a_{i+1,j+1}^{(k-1)}} [a_{ij}^{(k)} a_{i+1,j+1}^{(k)} - a_{i,j+1}^{(k)} a_{i+1,j}^{(k)}] \tag{6}$$

for $i, j=1, \cdots, n-k-1$; with $a_{ij}^{(-1)}=1$, $a_{ij}^{(0)}=a_{ij}$. It is found that $A_{n-1}=a_{11}^{(n-1)}$ $=\det A$.

As pointed out in [1], the correctness of the method may be deduced from a theorem regarding symmetrical rearrangements within determinants, as described, for example, by Aitken [2].

A count of arithmetical operations reveals that $(1/3)(n-1)n(2n-1)$ multiplications, $(1/6)(n-2)(n-1)(2n-3)$ divisions, and approximately $(1/4)(n-1)$ $n(2n-1)$ additions are required to perform the entire contraction. The number of multiplications is thus identical with that required for the pivotal condensation method [3]. As equations (6) indicate, a symmetric matrix A is thus contracted into a sequence of symmetric matrices A_k, so that in this case the computational labor is approximately halved.

A point requiring further elaboration concerns the necessary removal of "interior" zeros, *i.e.*, zero elements $a_{ij}^{(k)}$ that may occur anywhere in A_k except in the outside rows or columns.

It is easy to devise appropriate techniques for the removal of such zeros. Thus one may proceed as follows:

I. *Removal of interior zeros from the first row.* Let us first examine the first row R_1 of A. If no zeros occur in R_1, we may proceed to step II. If, on the other hand, there are zeros in R_1, let the first zero element in R_1 be a_{ij}, $j \leqq n-1$. Then we examine the column C_j of A. Let the first nonzero element in C_j be a_{ij}, $i \leqq n-1$. Consider now the $j-2$ equations for the parameter λ_k:

$$\lambda_k a_{1k} + a_{ik} = 0, \qquad k = 2, \cdots, j-1.$$

Thus $\lambda_k = -a_{ik}/a_{1k}$, for these k. Among these λ_k let the nonvanishing ones be denoted by $\mu_1, \cdots, \mu_s$. Then we put $\Lambda_1 \neq \mu_1^{-1}, \cdots, \mu_s^{-1}$, and replace in A the row R_1 by

$$R_1^{(1)} = R_1 + \Lambda_1 R_i. \tag{7}$$

Clearly then $a_{1k}^{(1)} \neq 0$ for $k = 2, \cdots, j$, so that the string of nonvanishing elements in the first row $R_1^{(1)}$ has been extended by at least one element beyond that in R_1. Certainly, $\det(R_1^{(1)}, R_2, \cdots, R_n) = \det(R_1, R_2, \cdots, R_n) = \det A$. This procedure is continued in the same manner until all the interior zero elements have been removed from the first row of the matrix; let this row be denoted by $R_1^{(s)}$, $s \geqq 1$.

II. *Removal of interior zeros from* $R_2, R_3, \cdots, R_{n-1}$. Assume, now, that in R_i, $i = 2, \cdots, n-1$, the elements $a_{ik} \neq 0$, $k = 2, \cdots, r-1$, but that $a_{ir} = 0$, with $r \leqq n-1$. As above, consider

$$\rho_k a_{1k}^{(s)} + a_{ik} = 0, \qquad 2 \leqq k \leqq r.$$

Thus $\rho_k = -a_{ik}/a_{1k}^{(s)}$. Choosing $\rho \neq \rho_k$, $2 \leqq k \leqq r$, replace R_i by

$$R_i^{(1)} = R_i + \rho R_1^{(s)}. \tag{8}$$

Now $a_{1k}^{(1)} \neq 0$ for $2 \leqq k \leqq r$, extending the string of nonzero elements in $R_i^{(1)}$ by at least one element. Further,

$$\det(R_1^{(s)}, R_2, \cdots, R_i, \cdots, R_n) = \det(R_1^{(s)}, R_2, \cdots, R_i^{(1)}, \cdots, R_n) = \det A.$$

In this manner there will be obtained a matrix $A^{(t)}$ which contains no interior zero elements, and for which $\det A^{(t)} = \det A$.

III. *Occurrence of interior zeros in* A_k. It may happen that in A_k a particular element of R_i vanishes, $2 \leqq i, j \leqq n-k-2$, $0 \leqq k \leqq n-4$.

In such a case one may simply return to $A^{(t)}$, replace R_i by $R_i + \sigma R_1$, and start the calculation over again, choosing the parameter σ in such a manner that the corresponding element in the transformed matrix A_k becomes nonzero while the nonvanishing of the previous interior elements remains preserved. Obviously a change of R_i in $A^{(t)}$ will in general produce changes of R_{i-k}, $R_{i-k+1}, \cdots, R_i$ in A_k.

The following example illustrates the procedure. Suppose it is desired to calculate $\det A$, where

$$A = \begin{bmatrix} 1 & -1 & 2 & 0 & 0 \\ -1 & 0 & 3 & 0 & 2 \\ 0 & 3 & -2 & 1 & -1 \\ 0 & -3 & -11 & -2 & 0 \\ 2 & 0 & 1 & 0 & 2 \end{bmatrix}.$$

Putting first $R_1^{(1)} = R_1 + 2R_3$, then $R_2^{(1)} = R_2 + R_1^{(r)}$, there results

$$A^{(1)} = \begin{bmatrix} 1 & 5 & -2 & 2 & -2 \\ 0 & 5 & 1 & 2 & 0 \\ 0 & 3 & -2 & 1 & -1 \\ 0 & -3 & -11 & -2 & 0 \\ 2 & 0 & 1 & 0 & 2 \end{bmatrix}.$$

It follows that

$$A_1 = \begin{bmatrix} 5 & 15 & -6 & 4 \\ 0 & -13 & 5 & -2 \\ 0 & -39 & 15 & -2 \\ 6 & -3 & 2 & -4 \end{bmatrix},$$

and

$$A_2 = \begin{bmatrix} -13 & -3 & -4 \\ 0 & 0 & -40 \\ 78 & 3 & 28 \end{bmatrix}.$$

Here the element $a_{22}^{(2)} = 0$. By III we may remove this defect by starting afresh, replacing in $A^{(1)}$ the row R_2 by $R_2 + \sigma R_1$. For $\sigma = 1$ there results

$$A^{(3)} = \begin{bmatrix} 1 & 5 & -2 & 2 & -2 \\ 1 & 10 & -1 & 4 & -2 \\ 0 & 3 & -2 & 1 & -1 \\ 0 & -3 & -11 & -2 & 0 \\ 2 & 0 & 1 & 0 & 2 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 5 & 15 & -6 & 4 \\ 3 & -17 & 7 & -2 \\ 0 & -39 & 15 & -2 \\ 6 & -3 & 2 & -4 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -13 & -3 & -4 \\ -39 & -9 & 16 \\ -78 & 3 & 28 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & -12 \\ 21 & -20 \end{bmatrix},$$

$$A_4 = \det A = 28.$$

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INVERSE ELLIPTIC FUNCTIONS AND LEGENDRE POLYNOMIALS

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Series expansions for the complete elliptic integrals $K(k)$ and $E(k)$ which involve Legendre polynomials have been known for quite some time. However, most of these expansions involve the Legendre polynomial of argument $\cos\theta$ (θ real), and here we shall obtain expansions of $K(k)$ in which the argument of the Legendre polynomials which appear is greater than one. The series for $K(k)$ appear as special cases of power series expansions of three of the Jacobian inverse elliptic functions: $\mathrm{sn}^{-1}(x, k)$, $\mathrm{cn}^{-1}(x, k)$ and $\mathrm{tn}^{-1}(x, k)$.* Throughout the following $K(k)$ denotes the complete elliptic integral of the first kind with modulus k, $0<k<1$. The complementary modulus k' is, as usual, defined to be $\sqrt{(1-k^2)}$.

If n is a nonnegative integer, $0<|x|\leqq 1$, $\lambda=\frac{1}{2}(x+x^{-1})$, and $P_n(\lambda)$ denotes the nth Legendre polynomial, then we define the polynomial $R_n(x)$ to be $x^nP_n(\lambda)$. Using Laplace's first integral for $P_n(\lambda)$ [4], we obtain after some simplification

$$R_n(x) = \frac{2}{\pi}\int_0^{\pi/2} (x^2\sin^2\theta + \cos^2\theta)^n d\theta > 0. \tag{1}$$

On expanding the integrand and integrating we have

$$R_n(x) = \frac{1}{2^{2n}}\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}x^{2k}, \tag{2}$$

but we shall not need this explicit expression for $R_n(x)$.

The Jacobian inverse elliptic function $\mathrm{sn}^{-1}(y, k)$ is defined by the integral

$$\mathrm{sn}^{-1}(y, k) = \int_0^y [(1-t^2)(1-k^2t^2)]^{-1/2}dt$$

if $|y|\leqq 1$ and $0<k<1$. Differentiating with respect to y and setting $ky^2=h$ and $\lambda=\frac{1}{2}(k+k^{-1})$ we have

$$\frac{d}{dy}\mathrm{sn}^{-1}(y, k) = (1-2\lambda h+h^2)^{-1/2} \text{ if } |y| < 1.$$

From the theory of Legendre polynomials [3]: If $F(z)=z\pm\sqrt{(z^2-1)}$, then

$$(1-2zh+h^2)^{-1/2} = \begin{cases} \sum_{n=0}^{\infty} P_n(z)h^n, & \text{if } |h| < \min|F(z)|, \\ \sum_{n=0}^{\infty} P_n(z)h^{-(n+1)}, & \text{if } |h| > \max|F(z)|. \end{cases} \tag{3}$$

* Here we are following the notation used in [2]; some books write $\mathrm{sc}^{-1}(x, k)$ rather than $\mathrm{tn}^{-1}(x,k)$.

Since $\min |F(\lambda)| = k$ it follows that

$$\frac{d}{dy}\,\mathrm{sn}^{-1}(y, k) = \sum_{n=0}^{\infty} P_n(\lambda)k^n y^{2n} \text{ if } |y| < 1.$$

Integrating both sides of this equation, $0 \leqq y \leqq x < 1$, we obtain

$$\mathrm{sn}^{-1}(x, k) = \sum_{n=0}^{\infty} P_n(\lambda)k^n \frac{x^{2n+1}}{2n+1}, \qquad |x| < 1. \tag{4}$$

Now $k^n P_n(\lambda) = R_n(k) > 0$ and by a theorem due to Pringsheim on the converse of Abel's Theorem ([1], p. 256) it follows that (4) is valid for $x = 1$ and hence $|x| \leqq 1$.

If we take $x = 1$ in (4) we obtain the special case

$$K(k) = \sum_{n=0}^{\infty} P_n(\lambda) \frac{k^n}{2n+1}.$$

If we take $x = (1+k')^{-1/2}$ and use the fact that $\mathrm{sn}^{-1}((1+k')^{-1/2}, k) = \frac{1}{2}K(k)$ ([2], p. 31), then we have

$$\sqrt{(1+k')}K(k) = 2\sum_{n=0}^{\infty} P_n(\lambda)\left(\frac{1-k'}{1+k'}\right)^{n/2} \frac{1}{2n+1}.$$

Expansions for the other two inverse elliptic functions are obtained similarly but conditions for the convergence of the series are more complicated. Differentiation of the integral

$$\mathrm{cn}^{-1}(y, k) = \int_y^1 [(1-t^2)(k'^2 - k^2t^2)]^{-1/2}dt, \qquad \text{if } |y| \leqq 1,$$

with respect to y together with the substitutions $ky^2 = ik'h$ and $\mu = \frac{1}{2}(k'/k - k/k')$ yields

$$\frac{d}{dy}\,\mathrm{cn}^{-1}(y, k) = -\frac{1}{k'}(1 - 2i\mu h + h^2)^{-1/2}, \qquad \text{if } |y| < 1.$$

Now

$$\min |F(i\mu)| = \begin{cases} k/k' & \text{if } 0 < k \leqq 1/\sqrt{2}, \\ k'/k & \text{if } 1/\sqrt{2} < k < 1, \end{cases}$$

so from (3) we have that

$$\frac{d}{dy}\,\mathrm{cn}^{-1}(y, k) = -\frac{1}{k'}\sum_{n=0}^{\infty} P_n(i\mu)(-i)^n(k/k')^n y^{2n}. \tag{5}$$

The series converges and represents the function on the left if (1) $0 < k \leqq 1/\sqrt{2}$ and $|y| < 1$, or if (2) $1/\sqrt{2} < k < 1$ and $|y| < k'/k$. From Laplace's first integral

for the Legendre polynomials it is easily shown that $P_n(i\mu)=i^nS_n(\mu)$ where $S_n(\mu)$ is a polynomial of degree n in the symbol μ. These polynomials satisfy the recurrence relation

$$(n+1)S_{n+1}(\mu) = (2n+1)\mu S_n(\mu) + nS_{n-1}(\mu),$$

from which it is seen that $S_n(\mu)>0$ if $\mu>0$. Clearly, $S_n(\mu)$ is the polynomial obtained by changing all minus signs in $P_n(\mu)$ to plus signs.

If $0<k\leqq 1/\sqrt{2}$ and we integrate (5), $0\leqq y\leqq x<1$, we have

$$\text{(6)} \qquad \mathrm{cn}^{-1}(x,k) - K(k) = -\frac{1}{k'}\sum_{n=0}^{\infty} S_n(\mu)(k/k')^n \frac{x^{2n+1}}{2n+1}.$$

Since $S_n(\mu)>0$, (6) is valid for $0\leqq x\leqq 1$ by the theorem of Pringsheim noted above. Also, it is clear that (6) is valid for $|x|\leqq 1$. If $1/\sqrt{2}<k<1$, then (6) is valid for $0\leqq x<k'/k$.

If $0<k\leqq 1/\sqrt{2}$ and $x=1$ we have

$$k'K(k) = \sum_{n=0}^{\infty} S_n(\mu)(k/k')^n \frac{1}{2n+1},$$

and if $x=\sqrt{\{k'/(1+k')\}}=\sqrt{(k'/k)}\sqrt{\{(1-k')/k\}}$, then from the relation $\mathrm{cn}^{-1}(\sqrt{\{k'/(1+k')\}}, k)=\frac{1}{2}K(k)$ ([2], p. 31) we obtain

$$K(k) = \frac{2}{k}\sqrt{\left(\frac{1-k'}{k'}\right)}\sum_{n=0}^{\infty} S_n(\mu)\left(\frac{1-k'}{k}\right)^n \frac{1}{2n+1}.$$

In the case of $\mathrm{tn}^{-1}(y,k)$ we obtain two series, one of which is a generalization of Gregory's series for $\tan^{-1}x$. Since

$$\mathrm{tn}^{-1}(y,k) = \int_0^y [(1+t^2)(1+k'^2t^2)]^{-1/2}dt, \qquad \text{if } 0\leqq y\leqq \infty,$$

differentiation of the integral together with the substitutions $k'y^2=h$ and $\lambda'=\frac{1}{2}\{k'+(1/k')\}$ yields

$$\frac{d}{dy}\mathrm{tn}^{-1}(y,k) = [1-2(-\lambda')h+h^2]^{-1/2}.$$

Now $\min|F(-\lambda')|=k'$ and $\max|F(-\lambda')|=1/k'$ so from (3) we have

$$\text{(7)} \qquad \frac{d}{dy}\mathrm{tn}^{-1}(y,k) = \sum_{n=0}^{\infty} P_n(\lambda')k'^n(-1)^n y^{2n} \text{ if } |y|<1,$$

and

$$\text{(8)} \qquad \frac{d}{dy}\mathrm{tn}^{-1}(y,k) = \sum_{n=0}^{\infty} \frac{P_n(\lambda')(-1)^n}{k'^{n+1}y^{2n+2}} \text{ if } |y|>1/k'.$$

Integrating (7), $0\leqq y\leqq x<1$, we have

$$\text{(9)} \qquad \mathrm{tn}^{-1}(x, k) = \sum_{n=0}^{\infty} P_n(\lambda')k'^n(-1)^n \frac{x^{2n+1}}{2n+1}.$$

If $0 \leqq x \leqq 1$ the series (9) is absolutely convergent and represents $\mathrm{tn}^{-1}(x, k)$ since the series obtained for $\mathrm{sn}^{-1}(x, k')$ is convergent in the same interval. The series on the right side of (8) converges uniformly for $1/k' \leqq y \leqq x_0$; integrating (8), $1/k' \leqq x \leqq y \leqq \infty$, we find*

$$\text{(10)} \qquad \mathrm{tn}^{-1}(x, k) = K(k) - \sum_{n=0}^{\infty} \frac{P_n(\lambda')(-1)^n}{k'^{n+1}x^{2n+1}} \frac{1}{2n+1} \quad \text{if} \quad x \geqq 1/k'.$$

Formally, if $k=0$ then (10) implies the expansion of $\tan^{-1} x$ known as Gregory's series for $\tan^{-1} x$. If we take $x=1/k'$ in (10) then we have

$$\text{(11)} \qquad \mathrm{tn}^{-1}(1, k) = \sum_{n=0}^{\infty} P_n(\lambda')k'^n(-1)^n \frac{1}{2n+1}$$

since $\mathrm{tn}^{-1}(1/k', k) + \mathrm{tn}^{-1}(1, k) = K(k)$ ([2], p. 32). Formally, if $k=0$ then $k'=1$, $P_n(\lambda')k'^n = R_n(1) = 1$, $\mathrm{tn}^{-1}(1, 0) = \pi/4$, and from (11) we obtain a well-known series for $\pi/4$.

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THE DIRAC MEASURE AS APPLIED TO THE SOLUTION OF DIFFERENCE EQUATIONS BY MEANS OF TRANSFORMS

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It is known that transforms (Laplace, Fourier, *etc.*) can be used to solve difference equations.§ In this note it is shown that by aid of the Dirac measure of L. Schwartz† (in essence the integral of Stieltjes) one can obtain the solution of linear difference equations.

Let z_k, $k=1, \cdots, n$ be a set of complex points on a contour C, and let $d\mu = a_k$ at $z=z_k$, $k=1, \cdots, n$, $d\mu=0$ elsewhere. We define the Stieltjes' integral

* For a justification of this step see [1] p. 500.

§ H. Bremmer and B. van der Pol, Operational Calculus, Cambridge, 1950; G. Doetsch, Handbuch der Laplace-Transformation, vol. 3, Basel and Stuttgart, 1950.

† L. Schwartz, Théorie des Distributions, tome 1, Paris, 1950.

of $f(z)$ along C by

$$\int_C f(z)d\mu \equiv \sum_{k=1}^{n} a_k f(z_k). \tag{1}$$

As a special case with $d\mu=1$ at $z=0$, $d\mu=0$ otherwise (the Dirac-measure of L. Schwartz), one has $\int_{-\infty}^{\infty} f(x)d\mu=f(0)$.

One can obtain a generalization of (1) as follows: Let $y=f(x)$, $-\infty<x<\infty$, $f(x)$ single-valued, represent a curve in the xy-plane, and define the measure $d\mu(x, y)$ by

$$\begin{aligned} d\mu(x, y) &= \rho(x) \qquad \text{for } y = f(x), \\ d\mu &= 0 \qquad \text{otherwise.} \end{aligned} \tag{2}$$

We define the integral of $g(x, y)$ relative to $d\mu(x, y)$ by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y)d\mu \equiv \int_{-\infty}^{\infty} g(x, f(x))\rho(x)dx \tag{3}$$

provided the Riemann-integral on the right of (3) exists. In particular,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (y - f(x))F(x, y)d\mu = 0 \tag{4}$$

for $d\mu$ defined by (2), with $\rho(x)$ arbitrary.

We now make use of these definitions to find the solutions of three simple homogeneous difference equations.

Let us determine a general solution of

$$H_{\nu-1}(z) + H_{\nu+1}(z) = -2H'_\nu(z). \tag{5}$$

We define $H_\nu(z)$ by

$$H_\nu(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\nu z}e^{-zy}d\mu \tag{6}$$

with $d\mu$ unspecified. Equation (5) yields (formally)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\nu z}e^{-zy}(\cosh x - y)d\mu = 0. \tag{7}$$

From (4) it follows that $d\mu=\rho(x)$ for $y=\cosh x$, $\rho(x)$ arbitrary, $d\mu=0$ elsewhere, is a solution of (7). The definition (3) along with (6) yields

$$H_\nu(z) = \int_{-\infty}^{\infty} \rho(x)e^{-\nu z}e^{-z\cosh x}dx = \int_{-\infty}^{\infty} \sigma(x)\cosh \nu x\, e^{-z\cosh x}dx \tag{8}$$

with $\sigma(x)$ arbitrary, provided the integral of (8) exists. For $\sigma(x)\equiv 1$ one obtains the integral representation of the modified Hankel function,

$$K_\nu(z) = \int_{-\infty}^{\infty} \cosh \nu x \, e^{-z \cosh x} dx, \qquad \text{for } \operatorname{Re} z > 0.$$

A simple generalized random walk problem leads to the equation

$$\phi_{n+1}(z) = \int_{-\infty}^{\infty} \phi_n(z - t)\phi(t)dt \tag{9}$$

with $\int_{-\infty}^{\infty}\phi(t)dt=1$, $\phi_1(z)=\phi(z)$. The Fourier integral of $\phi(t)$ is given by $\psi(y)=\int_{-\infty}^{\infty}\phi(t)e^{iyt}dt$.

We define $\phi_n(z)$ by

$$\phi_n(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-nz}e^{-iyz}d\mu \tag{10}$$

with $d\mu$ as yet undefined. Quite formally (9) and (10) yield

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-nz}e^{-iyz}[e^{-z} - \psi(y)]d\mu = 0. \tag{11}$$

A solution of (11) for $d\mu$ is given by $d\mu=\rho(y)$ for $e^{-z}=\psi(y)$, $d\mu=0$ elsewhere. Thus (10) yields

$$\phi_n(z) = \int_{-\infty}^{\infty} \rho(y)[\psi(y)]^n e^{-iyz}dy. \tag{12}$$

Since

$$\phi(z) = \phi_1(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(y)e^{-izy}dy,$$

it follows that $\rho(y)\equiv 1/2\pi$ so that

$$\phi_n(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [\psi(y)]^n e^{-iyz}dy. \tag{13}$$

As an application of (1) we consider $f(s+1)+f(s)=0$. Let $f(s)=\int_C e^{-sz}d\mu$ with $d\mu$ and the contour C as yet unspecified. It follows that $\int_C e^{-sz}(e^{-z}+1)d\mu=0$, so that $d\mu=a_k$, a_k arbitrary, for z_k satisfying $e^{-z_k}+1=0$. Thus $z_k=-\pi i(1+2k)$, $k=0, \pm 1, \pm 2, \cdots$, and the contour C is chosen as the imaginary axis. It follows that

$$f(s) = \sum_{k=-\infty}^{\infty} a_k e^{\pi i s(1+2k)} \tag{14}$$

provided the series converges.

Since the method outlined above requires formal manipulations, one must verify that the solution so obtained does satisfy the difference equation in question.

THEOREMS ON NONASSOCIATIVE NUMBER THEORY

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The *free logarithmetic* $\mathfrak{L}$ can be defined as the arithmetic of the indices of powers of the generator of a free cyclic groupoid.* *Indices of* $\mathfrak{L}$ are analogous to Etherington's noncommutative *shapes* [1], his *partitioned serials* [2] and his *indices* of the "most general" logarithmetic B [3], to elements of Robinson's *simple forest* [6] and to Evans' *nonassociative numbers* [4]. Systems of postulates for $\mathfrak{L}$ have been given by Etherington [2], Robinson [6] and Evans [4]. Some basic results in the nonassociative number theory, notably the unique factorization theorem, have been obtained by Etherington [2], Evans [4] and the author [5].

Addition and multiplication in $\mathfrak{L}$ are both noncommutative. Two indices commute additively if and only if they are equal. In Theorem 1 we give a necessary and sufficient condition that two indices of $\mathfrak{L}$ should commute multiplicatively. In Theorems 2 and 3 we solve Diophantine-like equations. In [4] Evans has proved "Fermat's Last Theorem" for nonassociative numbers. Theorem 3 generalizes this result which is then deduced as a corollary. We use the notation and nomenclature of [5].

It is convenient to extend the definition of exponentiation of an index (*cf.* [2], p. 449): We define $P^0=1$ for all indices P of $\mathfrak{L}$.

THEOREM 1. *P and Q, two indices of $\mathfrak{L}$, commute with respect to multiplication, i.e., $PQ=QP$, if and only if they are powers of the same index.*

Proof. The condition is obviously sufficient. To prove necessity denote the number of prime factors in an index X by $pr(X)$ and use induction on $pr(PQ)$. If $pr(PQ)=0$ or 1 then either P or Q is equal to 1 and either $P=Q^0$ or $Q=P^0$. Now let $pr(PQ)=a$ $(a>1)$ and assume that the condition is necessary for all pairs of commuting indices whose products contain less than a prime factors. Without loss of generality we can suppose that $pr(P)\leq pr(Q)$. If $P=1$ then $P=Q^0$. Otherwise $PQ=QP$ gives $Q=PQ'$, where Q' is a proper left-divisor of Q or is equal to 1. It follows that $P^2Q'=PQ'P$ and therefore $PQ'=Q'P$. Now, $pr(PQ')<a$ and, by the induction hypothesis, $P=R^s$ and $Q'=R^t$ for some index R. Hence $P=R^s$ and $Q=R^{s+t}$.

THEOREM 2. *If $X^mP=Y^n$ or $PX^m=Y^n$, where X, Y, P are indices of $\mathfrak{L}$, P is prime and m, n are integers greater than 1, then X and Y are both powers of P.*

Proof. $X^mP=Y^n$ implies either (i) $X=Y^sY'$ or (ii) $Y=X^tX'$, where s and t are maximal in the sense that Y', X' are proper left-divisors of Y, X respectively or are equal to 1.

If (i) $X=Y^sY'$ then $XP=Y^sY'P$ and, since XP is a proper right-divisor of

* *Cf.* [5]. A *groupoid* is a set closed with respect to a binary product operation. It is *cyclic* if it is generated by one element.

Y^a and $pr(Y^a) < pr(XP) \leqq pr(Y^{a+1})$, $XP = Y''Y^a$ where Y'' is a proper right-divisor of Y or is equal to Y. But $pr(Y'') = pr(Y'P) = pr(XP) - pr(Y^a)$ and since Y'' is a left-divisor of XP it is a left-divisor of X and thus of Y. Hence $Y'' = Y'Q$ where Q is a prime index. Now, Y'' is also a right-divisor of Y so that Q is the prime right-divisor of Y. Therefore $Q = P$ and $Y^aY'P = Y'PY^a$, *i.e.*, Y^a and $Y'P$ commute. Hence, by Theorem 1, $Y^a = R^a$ and $Y'P = R^b$ for some index R. Note that P is a right-divisor of R and R is a left-divisor of Y and let $R = Y_1 \cdots Y_{k-1}P$ where the Y_i are prime. Then

$$XP = (Y_1 \cdots Y_{k-1}P)^{a+b}, \qquad X = (Y_1 \cdots Y_{k-1}P)^{a+b-1}Y_1 \cdots Y_{k-1}$$

and, as $X^mP = Y^a$ and $m > 1$,

$$(Y_1 \cdots Y_{k-1}P)^{a+b-1}Y_1 \cdots Y_{k-1}(Y_1 \cdots Y_{k-1}P)^{a+b-1} \cdots P = (Y_1 \cdots Y_{k-1}P)^g \cdots,$$

where $g \geqq 2a + 2b - 1$. Since factorization into primes is unique in $\mathfrak{L}$ the $(k(a+b))$th, $(k(a+b)+1)$th, $\cdots$, $(k(a+b)+k-1)$th prime factors on both sides are equal, *i.e.*, $Y_1 = P$, $Y_2 = Y_1$, $\cdots$, $P = Y_{k-1}$. Hence $Y_1 = \cdots = Y_{k-1} = P$ and X and Y are both powers of P.

If (ii) $Y = X^tX'$ consider first the case when $X' = 1$. Then $X^mP = X^{at}$. Hence $P = X^{at-m}$ and since P is prime $X = P$ and $Y = P^t$. If $X' \neq 1$ it is a proper left-divisor of X. Since P is a right-divisor of Y it is a right-divisor of X'. Let $X' = X''P$. Now, $pr(X^t) < pr(Y) < pr(X^{t+1})$ and Y is a right-divisor of X^mP; therefore $Y = X'''X^tP$, where X''' is a proper right-divisor of X or is equal to 1. But X''' is a left-divisor of Y and therefore of X. Also X'' is a left-divisor of X and $pr(X'') = pr(X''')$. It follows that $X'' = X'''$ and $Y = X^tX''P = X''X^tP$. Hence $X^tX'' = X''X^t$, *i.e.*, X^t and X'' commute and, by Theorem 1, $X^t = (X_1 \cdots X_h)^c$ and $X'' = (X_1 \cdots X_h)^d$ where the X_i are prime. Remembering that $n > 1$ we have

$$(X_1 \cdots X_h)^f \cdots P = (X_1 \cdots X_h)^{c+d}P(X_1 \cdots X_h)^{c+d} \cdots,$$

where $f \geqq 2c + 2d$. Comparing the $(h(c+d)+1)$th, $(h(c+d)+2)$th, $\cdots$, $(h(c+d)+h)$th prime factors on both sides we have $X_1 = P$, $X_2 = X_1$, $X_3 = X_2$, $\cdots$, $X_h = X_{h-1}$. Hence $X_1 = \cdots = X_h = P$ and both X and Y are powers of P.

If $PX^m = Y^a$ the proof is similar.

Note. In the statements and proofs of Theorems 1 and 2 no direct use is made of the operation of addition. It follows that these theorems are really about the free *multiplicative* logarithmetic $\mathfrak{L}^\times$, the semigroup formed by all indices and their multiplication (*v.* [5], p. 321).

I am indebted to a referee for drawing my attention to the fact that Theorems 1 and 2 actually apply to free semigroups. Indeed the free semigroup with ξ generators is isomorphic to any subsemigroup of $\mathfrak{L}^\times$ generated by ξ prime indices.

THEOREM 3. *If* $X^p+Y^q=Z^r$ *where* X, Y, Z *are indices of* $\mathfrak{P}$ *and* p, q, r *are integers greater than* 1 *then* $X=2^k$, $Y=2^m$, $Z=2^n$ *and* $kp=mq=nr-1$.

Proof. Let $Z=Z_1\cdots Z_n$, where the Z_i are prime. $X^p+Y^q=Z^r$ implies $X^p=(Z_1\cdots Z_n)^{r-1}Z_1\cdots Z_{n-1}Z_n'$ and $Y^q=(Z_1\cdots Z_n)^{r-1}Z_1\cdots Z_{n-1}Z_n''$ where $Z_n'+Z_n''=Z_n$ (*Cf.* [4], p. 302 or [5], p. 339). Since Z_n is prime, Z_n' and Z_n'' must be mutually left-prime.

Suppose Z_n', $Z_n''\neq 1$. Then $X=UX'$ and $Y=UY'$ where X', Y' are mutually left-prime and neither is 1. Therefore $U(X'X^{p-1}+Y'Y^{q-1})=Z^r$ and $X'X^{p-1}+Y'Y^{q-1}$ is prime. Since p, $q>1$ the potency* of $X'X^{p-1}+Y'Y^{q-1}$ is greater than that of U and hence $X'X^{p-1}+Y'Y^{q-1}$ is not a factor of U. Thus a prime occurs only once as a factor of X^p+Y^q. But every prime factor of Z^r occurs at least r times, *i.e.*, more than once and $X^p+Y^q=Z^r$ which is a contradiction. Therefore either Z_n' or Z_n'' is equal to 1.

If Z_n' is equal to 1, $X^pZ_n=Z^r$, where Z_n is prime and p, $r>1$. Therefore, by Theorem 2, $X=Z_n^k$, $Z=Z_n^n$. Thus $Y^q=Z_n^{nr-1}Z_n''$. Since $q>1$ the potency of Y is not greater than half of that of $Z_n^{nr-1}Z_n''$. Now, $nr-1>0$ and the potency of Z_n'' is less than that of Z_n, which is prime. Hence all prime factors of Y are equal to Z_n and $Z_n''=1$. Therefore $Z_n=Z_n'+Z_n''=2$ and $X=2^k$, $Y=2^m$, $Z=2^n$ where $kp=mq=nr-1$.

If $Z_n''=1$ the proof is similar.

COROLLARY ("Fermat's Last Theorem"). $X^r+Y^r=Z^r$ *implies* $r=1$.

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* The *potency* of an index P, δ_P, (*v.* [5], p. 324) corresponds to Etherington's *degree* of the index P, δ_P, and to Evans' *length* of the nonassociative number P, $|P|$.

MATHEMATICAL NOTES

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EXTENSIONS OF AN INEQUALITY OF H. S. SHAPIRO

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H. S. Shapiro's inequality [1] is

$$\sum \frac{x_i}{x_{i+1}+x_{i+2}} \geqq \frac{n}{2}, \tag{1}$$

where (and throughout this note) $x_{n+j}=x_j>0$ for all j, i is the summation suffix, and i runs over 1 to n unless otherwise indicated. This has been proved true for $n=3, 4, 5$ and false for $n=20$ (see [1]). It is also trivially true for $n=1, 2$. I extend it to

$$\sum \frac{x_i}{x_{i+1}+\cdots+x_{i+m}} \geqq \frac{n}{m}, \tag{2}$$

where now each denominator on the left consists of m terms, and prove the following results.

THEOREM 1. *The inequality* (2) *is true if*

$$\sin\frac{r\pi}{n} \geqq \sin(2m+1)\frac{r\pi}{n} \qquad \left(r=1,\cdots,\left[\frac{n}{2}\right]\right). \tag{3}$$

THEOREM 2. *The inequality* (2) *is true if*

$$n \mid m+2 \text{ or } 2m \text{ or } 2m+1 \text{ or } 2m+2. \tag{4}$$

It follows easily from Theorem 2 that Shapiro's inequality (1) is true for $n\leqq 6$.

I first prove Theorem 1. The left hand side of (2) is the weighted sum of $1/(x_{i+1}+\cdots+x_{i+m})$ $(i=1,\cdots,n)$ with weights $x_1,\cdots,x_n$ respectively. Thus since a weighted arithmetic mean is not less than the corresponding weighted harmonic mean we have

$$\sum \frac{x_i}{x_{i+1}+\cdots+x_{i+m}} \geqq \frac{(\sum x_i)^2}{\sum\{x_i(x_{i+1}+\cdots+x_{i+m})\}}. \tag{5}$$

Hence Theorem 1 follows from

LEMMA 1. *If* (3) *is true,*

$$(\sum x_i)^2 \geqq \frac{n}{m}\sum\{x_i(x_{i+1}+\cdots+x_{i+m})\}. \tag{6}$$

Proof. The inequality (6) holds if and only if the matrix of the quadratic form

$$(\sum x_i)^2 - \frac{n}{m}\sum \{x_i(x_{i+1} + \cdots + x_{i+m})\}$$

is positive definite or semidefinite. This matrix is clearly a circulant. Its first row can be obtained by writing down a row of n ones and subtracting, in succession, $n/2m$ from the elements (possibly modified by previous subtraction) in positions $2, \cdots, m+1; n-m+1, \cdots, n$ (mod n). Now this matrix is positive definite or semidefinite if and only if all its characteristic roots are nonnegative. Since the matrix is a circulant, it is seen that these roots are $\lambda_1, \cdots, \lambda_n$ where

$$\lambda_r = \sum_1^n \omega_r^i - \frac{n}{2m}\sum_1^m (\omega_r^i + \omega_r^{-i}) \qquad (r = 1, \cdots, n),$$

with $\omega_r = \exp(2r\pi i/n)$. Clearly $\lambda_n = 0$. For $r < n$,

$$\lambda_r = -\frac{n}{2m}\left\{\frac{\omega_r^{m+1} - \omega_r}{\omega_r - 1} + \frac{\omega_r^{-m-1} - \omega_r^{-1}}{\omega_r^{-1} - 1}\right\}$$

$$= \frac{n}{2m} \cdot \frac{\sin\dfrac{r\pi}{n} - \sin(2m+1)\dfrac{r\pi}{n}}{\sin\dfrac{r\pi}{n}},$$

after some simplification. It is seen that $\lambda_{n-r} = \lambda_r$. Hence $\lambda_r \geqq 0$, for all relevant r, if and only if (3) is true. The proof of the lemma is now complete.

Theorem 2 now follows from Theorem 1 by virtue of

LEMMA 2. *If* (4) *is true,* (3) *is true.*

Proof. It is seen that (3) is equivalent to

$$\sin\frac{r\pi}{n} \geqq \left\{\begin{array}{ll} -\sin\dfrac{3r\pi}{n} & (\text{if } n \mid m+2) \\ \pm\sin\dfrac{r\pi}{n} & (\text{if } n \mid 2m \text{ or } 2m+2) \\ 0 & (\text{if } n \mid 2m+1) \end{array}\right\} \qquad \left(r = 1, \cdots, \left[\frac{n}{2}\right]\right).$$

The lemma clearly follows if we note that the first of these inequalities is equivalent to

$$\sin\frac{2r\pi}{n}\cos\frac{r\pi}{n} \geqq 0.$$

Remark 1. It is easily shown that (3) is false if $n > 2m+2$. That (3) is not

(for all m) necessary for the truth of (2) follows since, if $m=1$, (2) is true for all n.

Remark 2. In (3) the range for r can be reduced to $1, \cdots, [n/2]-1$, the inequality corresponding to $r=[n/2]$ being always satisfied.

The proof of Theorem 1 can be modified to prove

THEOREM 3. *The inequality*

$$\sum \frac{a_1x_{i+1}+\cdots+a_nx_{i+n}}{b_1x_{i+1}+\cdots+b_nx_{i+n}} \geqq \frac{n\sum a_i}{\sum b_i}, \tag{7}$$

where the numerators and denominators are all positive, is true if

$$(\sum a_i\omega_r^i)(\sum b_i\omega_r^{-i})+(\sum a_i\omega_r^{-i})(\sum b_i\omega_r^i) \leqq 0 \qquad \left(r=1,\cdots,\left[\frac{n}{2}\right]\right)$$

where $\omega_r=\exp(2r\pi i/n)$.

This extends Theorem 1, (7) being an extension of (2) and of Shapiro's inequality (1).

I am indebted to Dr. U. C. Guha and to Prof. A. Oppenheim for valuable discussions on these inequalities.

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GROUPS AS UNIONS OF PROPER SUBGROUPS

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1. Introduction. It is evident that any group G which is not monogenic (generated by a single element) is expressible as a union of proper subgroups; for example, such a G is the union of its monogenic subgroups, which by hypothesis are all proper. Conversely, if a group is a union of proper subgroups of itself, it clearly cannot be monogenic.

It is not so easy to characterize groups which are *finite* unions of their proper subgroups. That there exist nonmonogenic groups which are not such finite unions is seen from the example of the additive group Q^+ of rational numbers. Indeed, suppose Q^+ were the union of its proper subgroups $H_1, \cdots, H_n$. Since there are only a finite number of these H's, it can evidently be assumed that this union is irredundant—that is, that none of the H's is contained in the union of all the others. It follows that any of the H's, for example H_1, contains a rational $r=m/n$ which is in no other H_i. This implies that for all integers h, r/h can be in no H_i but H_1, and hence must be in H_1. This is true in particular for h a multiple of m, say $h=dm$, which makes $r/h=m/ndm=1/nd$. But if this is in H_1, so is any integral multiple of it, and in particular so is any cn-tuple of it. Since $cn\cdot 1/nd=c/d$ is a completely arbitrary rational, this means that H_1 is the whole of Q^+, a contradiction. (This proof shows, incidentally, that Q^+ can

never be an irredundant union of even infinitely many of its proper subgroups.)

Groups which *are* finite unions of their proper subgroups are the chief subject of the present note. Specifically, some conditions are derived which restrict the minimum number of proper subgroups into which a group can be decomposed.

2. Unions of two and three subgroups. We first prove

THEOREM 1. *No group is the union of two of its proper subgroups.*

Proof. If G is the union of its proper subgroups A and B, the union must be irredundant; that is, neither A nor B can contain the other. This being the case, let x be an element of A not contained in B, and y an element of B not contained in A. But if xy is in A then $x^{-1}xy=y$ is in A; similarly, if xy is in B, x must be in B. Thus in any case we have a contradiction.

The argument used to prove Theorem 1 can be applied to the proof of the following more general

LEMMA. *Let G be the irredundant union of the subgroups H_i. Then for each i, H_i contains the intersection of all the remaining H's.*

Proof. Since the union is irredundant, H_i cannot be contained in the union H of the remaining H's. Let x be an element of H_i which is contained in none of the other H's, and let y be an element contained in all the other H's (and so in their intersection). If xy is in H, it must be in some H_j; but since y is in every H_j, it follows that x is also in H_j; a contradiction. On the other hand, if xy is not in H, it must be in H_i; but since x is in H_i, this implies that y is in H_i, which proves the Lemma.

That a group may be a union of three proper subgroups is shown by the example of the Klein 4-group ($\{e, a, b, c\}$ with the relations $a^2=b^2=c^2=e$, $ab=ba=c$, $bc=cb=a$, $ca=ac=b$), which is the union of the three subgroups $\{e, a\}$, $\{e, b\}$, $\{e, c\}$. In a sense, in fact, the 4-group is characteristic of all groups which are unions of three proper subgroups. Specifically, we have

THEOREM 2. *A group G is the union of three proper subgroups if and only if the Klein 4-group is a homomorphic image of G.*

Proof. Since the inverse image of a proper subgroup under a homomorphism is a proper subgroup, the "if" part is clear by the preceding paragraph. To prove the "only if," suppose that G is the union of the proper subgroups A, B and C. It follows from Theorem 1 that this union must be irredundant; hence the sets $A'=A-(B\cup C)$, $B'=B-(C\cup A)$, $C'=C-(A\cup B)$ are all nonempty. On the other hand, by the Lemma, $A\cap B\subset C$, $B\cap C\subset A$, and $C\cap A\subset B$, so that $A\cap B\cap C=A\cap B=B\cap C=C\cap A$; call it H. Thus evidently G is the disjoint union of H, A', B' and C'. By arguments similar to those used above, it can be shown that, in fact, H is normal; A', B' and C' are cosets of H; and G/H is the 4-group.

3. An indecomposability criterion. In setting bounds on the smallness of the number of proper subgroups of which a group G can be the union, a useful tool is the following

LEMMA. *Let the group G be the irredundant union of subgroups A_i ($i=1, \cdots, n>2$) and set $M=A_2 \cup A_3 \cup \cdots \cup A_n$. Then if x is not in M, we have x^k in M for some $k=1, \cdots, n-1$.*

Proof. Clearly x is in A_1. Choose y in A_2 and not in A_1. Then $x^j y$ is in M for $j=1, \cdots, n-1$ since $x^j y$ in A_1 would yield y in A_1. If $x^j y$ is in A_2 for some $j=1, \cdots, n-1$, then x^j is in $A_2 \subset M$, as desired. If we have $x^j y = x^m y$ for some $j, m=1, \cdots, n-1$ with $j>m$, then $x^{j-m}=e$ is in M, as desired. Hence we may assume that the $n-1$ elements $xy, x^2y, \cdots, x^{n-1}y$ are distinct and in $A_3 \cup \cdots \cup A_n$. It follows that $x^j y, x^m y$ are in A_q for some $q=3, \cdots, n$ and some $j, m=1, \cdots, n-1$ with $j>m$. Then $x^{j-m}=(x^j y)(x^m y)^{-1}$ is in $A_q \subset M$, and the lemma is proved.

On the basis of this lemma we can now prove

THEOREM 3. *Suppose that kth roots can be taken in the group G for every positive integer k less than a certain n. Then G is not the irredundant union of n (or fewer!) of its proper subgroups.*

Proof. Since the hypothesis for the given n implies the analogous hypothesis for any m smaller than n, the "or fewer" part is an obvious consequence of the rest of the Theorem. Suppose, then, that G is the irredundant union of exactly n subgroups, and adopt the notation of the Lemma. If the element x is not in M, clearly no root y of x can be in M. This is true, for example, for $y=$ the $(n-1)!$th root of x, which exists in G by hypothesis. On the other hand, by the Lemma, y^k is in M for some k less than n. Since $x=(y^k)^r$, where $r=(n-1)!/k$, x too must be in M; a contradiction.

If G is a finite group of order N, the hypothesis of Theorem 3 is equivalent to the requirement that $(n-1)!$ be prime to N. (See, for example, this MONTHLY, vol. 60, 1953, pp. 185–6.) This gives us the immediate

COROLLARY. *Let G be a finite group of order N, p the smallest prime dividing N. Then G is not the union of p or fewer of its proper subgroups.*

The criterion of Theorem 3 cannot be strengthened. Indeed, let G be the abelian group generated by two elements x, y with the relations $x^p=y^p=e$; then G is the union of the $p+1$ proper subgroups generated by the elements $x, y, xy, x^2y, x^3y, \cdots, x^{p-1}y$, respectively.

For finite groups, the case in which (as in the example just given) the minimum imposed by Theorem 3 is actually assumed may be partially characterized by

THEOREM 4. *With notation as in the Corollary to Theorem 3, suppose that G is the union of exactly $p+1$ proper subgroups S_i; then at least one of the S's, say*

S_j, has index p. If, moreover, this S_j is normal, then all the S_i have index p and p^2 divides N.

Proof. If none of the S's has index p, they must all have indexes greater than p; hence for each i, $o(S_i) \leqq N/p+1$. Then we have $N < \sum o(S_i) < (p+1)$. $N/(p+1) = N$; a contradiction. Hence some S_j must have index p; assume now that this S_j is normal. Then for $i \neq j$, S_iS_j is a subgroup of G, which cannot be proper because G is the union of S_iS_j and the $p-1$ proper subgroups S_k for $k=1, \cdots, p+1$, $k \neq i$, $k \neq j$. Thus $S_iS_j = G$ for $i \neq j$, and, since S_j is normal, we have $o(G)o(S_i \cap S_j) = o(S_i)o(S_j)$, or $p(o(S_i \cap S_j)) = o(S_i)$ for $i \neq j$. Set $o(S_i) = N/q_i$ for $i \neq j$, so that $q_i \geqq p$. Suppose that $q_i > p$ for some $i \neq j$. Then

$$\begin{aligned} N = o(G) &\leqq o(S_j) + \sum_{i \neq j} [o(S_i) - o(S_i \cap S_j)] \\ &\leqq (N/p) + \sum_{i \neq j} [(N/q_i) - (N/pq_i)] \\ &< (N/p) + p((N/p) - (N/p^2)) = N; \end{aligned}$$

a contradiction. Hence $q_i = p$ for every $i \neq j$, and $o(S_i \cap S_j) = N/p^2$ shows that p^2 divides N.

ON THE NUMBER OF DISTINCT ZEROS OF POLYNOMIALS

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Although a polynomial of degree n must have n zeros, we are well aware of the fact that some or even all of them may coincide. Consequently, nothing much can be said about the number of distinct zeros of an arbitrary polynomial. If, however, we consider related pairs of polynomials, certain sharp estimates of this number can be obtained. Furthermore, from these very simply proven estimates, corollaries on a class of polynomial-Diophantine equations can be extracted which seem quite difficult to prove otherwise.

THEOREM 1. *If $P(x)$ is of degree n, $Q(x)$ is of degree m, and $n > m \geqq 0$, then there exist $n-m+1$ distinct x's for which either $P(x) = 0$ or $P(x) = Q(x)$.*

Remark. That the estimate $n-m+1$ is sharp is shown by the example $P(x) = x^n$, $Q(x) = x^m$.

Proof. Let $A(x)$, of degree r, be the g.c.d. of $P(x)$ and $Q(x)$ and form

$$p(x) = \frac{P(x)}{A(x)}, \qquad q(x) = \frac{Q(x)}{A(x)}.$$

Suppose that $p(x) = a_0(x-r_1)^{m_1} \cdots (x-r_k)^{m_k}$ and note that $p(x) - q(x)$ has no zeros in common with $p(x)$. It is a known theorem that the number of *distinct* zeros of a polynomial $B(x) = \deg B$—number of common zeros of B and B'. We apply this to $p(x) - q(x)$. We must, therefore, estimate the number of common zeros of $p(x) = q(x)$ and $p'(x) = q'(x)$. Since

$$\frac{p'(x)}{p(x)} = \frac{m_1}{x - r_1} + \cdots + \frac{m_k}{x - r_k},$$

it follows that the common roots must satisfy the equation

$$q(x)\left\{\frac{m_1}{x - r_1} + \cdots + \frac{m_k}{x - r_k}\right\} = q'(x).$$

This last equation when multiplied out has degree $\leqq \deg q' + k = m - r - 1 + k$. Hence, the number of common zeros is bounded by this quantity $m - r - 1 + k$. Consequently,

$$\text{number of distinct zeros of } p - q \geqq \deg (p - q) - m + r + 1 - k,$$

$$\text{number of distinct zeros of } p - q \geqq n - r - m + r + 1 - k,$$

$$\text{number of distinct zeros of } p - q \geqq n - m + 1 - k.$$

Since p and $p-q$ have no common zeros and p has k distinct zeros, it now follows that the number of distinct x's for which $p=0$ or $p=q$ is $\geqq n-m+1$. Still $\geqq$ to this quantity is the number of distinct x's for which $P=0$ or $P=Q$ and the theorem is proven.

The main corollary of this theorem is

THEOREM 2. *If $R(x)$ is a given polynomial of degree $c \geqq 0$, then $P^a(x) - Q^b(x) = R(x)$ is impossible for polynomials (nontrivial) P and Q with complex coefficients if either a or b exceeds $2c$, $a > 1$, $b > 1$.*

Proof. Suppose that the equality holds, then

$$\text{if } \deg P = n \neq 0 \text{ and } Q = m \neq 0, \deg P^a = na, \deg Q^b = mb.$$

We may assume without loss of generality that $na = mb$.

By Theorem 1, $P^a = 0$ or $P^a = R$ for $na - c + 1$ distinct values of x or equivalently $P^a = 0$ or $Q^b = 0$ for $na - c + 1$ distinct values of x. The number of such distinct values, however, is $\leqq n + m$. Consequently,

$$n + m \geqq na - c + 1,$$

$$n(a + b) \geqq nb + mb \geqq nab - cb + b,$$

$$(c - 1)b \geqq n[(a - 1)(b - 1) - 1] \geqq (a - 1)(b - 1) - 1,$$

$$a \leqq c + \frac{c}{b - 1} \leqq 2c.$$

Similarly, $b \leqq 2c$.

Another statement of this latter result is that $\deg \{P^a - Q^b\} \geqq \frac{1}{2} \max (a, b)$ unless P or Q is a constant or $P^a = Q^b$.

A special case of Theorem 2 is that $P^a(x) - Q^b(x) = x$, $a, b > 1$, has polynomial solutions only if $a = b = 2$ and in this case they are given by

$$P = \frac{k^2x + 1}{2k}, \qquad Q = \pm \frac{k^2x - 1}{2k}.$$

In a similar way, we can estimate quantities such as

$$\deg \{xP^a - (2x^2 + 3)Q^b\}, \qquad \deg \{P^aQ^b - R^cS^d\},$$

etc., and thereby obtain other results for polynomial-Diophantine equations.

GEOMETRY OF TRIHEDRONS*

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1. A necessary and sufficient condition for three lines $D\alpha$, $D\beta$, $D\gamma$, lying respectively in the faces DBC, DCA, DAB of a trihedron $(D) \equiv D - ABC$, to be coplanar is

$$\frac{\sin u}{\sin v} \cdot \frac{\sin u'}{\sin v'} \cdot \frac{\sin u''}{\sin v''} = +1, \tag{1}$$

where u, v, u', v', u'', v'' denote the angles $(D\alpha, DB)$, $(D\alpha, DC)$, $(D\beta, DC)$, $(D\beta, DA)$, $(D\gamma, DA)$, $(D\gamma, DB)$, respectively. This is a well-known theorem [1].

COROLLARY. *If the lines $D\alpha$, $D\beta$, $D\gamma$ are coplanar, then so are their isogonals $D\alpha'$, $D\beta'$, $D\gamma'$ with respect to the faces DBC, DCA, DAB.*

For, (1) implies

$$\frac{\sin v}{\sin u} \cdot \frac{\sin v'}{\sin u'} \cdot \frac{\sin v''}{\sin u''} = +1. \tag{2}$$

COROLLARY. *Let U and V, U' and V', U'' and V'' denote the dihedral angles formed by the planes $DA\alpha$, $DB\beta$, $DC\gamma$ in the dihedrals of (D) with the edges DA DB, DC, respectively. In order that the lines $D\alpha$, $D\beta$, $D\gamma$ be coplanar it is necessary and sufficient that*

$$\frac{\sin U}{\sin V} \cdot \frac{\sin U'}{\sin V'} \cdot \frac{\sin U''}{\sin V''} = +1. \tag{3}$$

In fact, the equivalence of (3) and (2) follows from that of the measures

$$DB \cdot D\alpha \sin u \cdot AA'/6 = 2DAC \cdot DA\alpha \sin U/3DA,$$

$$DC \cdot D\alpha \sin v \cdot AA'/6 = 2DAB \cdot DA\alpha \sin V/3DA,$$

of the volumes of the tetrahedrons $DAB\alpha$ and $DA\alpha C$, AA' being an altitude of the tetrahedron $DABC$, and so on.

COROLLARY. *If the lines $D\alpha$, $D\beta$, $D\gamma$ are coplanar, then so are the lines of intersection $D\alpha'$, $D\beta'$, $D\gamma'$ of the isogonal planes of $DA\alpha$, $DB\beta$, $DC\gamma$ with respect to*

* Translated by Bomshik Chang, University of British Columbia.

the dihedrals along the edges DA, DB, DC, with the planes of the faces DBC, DCA, DAB.

For (3) implies

$$\frac{\sin V}{\sin U}\cdot\frac{\sin V'}{\sin U'}\cdot\frac{\sin V''}{\sin U''} = +1. \tag{4}$$

2. Consequences. *If the isogonals, with respect to a tetrahedron $T\equiv ABCD$, of the lines AA', BB', CC', DD' which join the vertices of two tetrahedrons T and $T'\equiv A'B'C'D'$, each inscribed and circumscribed to the other, meet the planes of the faces of T (or T') opposite to A, B, C, D (or A', B', C', D') in A_1, B_1, C_1, D_1, then the tetrahedron $T_1\equiv A_1B_1C_1D_1$ is inscribed and circumscribed to T or(T').*

It suffices to show that the plane $B_1C_1D_1$ passes through A. Now, since the planes (AB, B') and (AB, B_1) are isogonal with respect to the dihedral AB, the dihedral angles which they form with the two faces of this dihedral are U and V for one and V and U for the other. Since the lines AB', AC', AD' are coplanar by hypothesis, it follows from the preceding corollary that

$$\frac{\sin U}{\sin V}\cdot\frac{\sin U'}{\sin V'}\cdot\frac{\sin U''}{\sin V''} = +1. \tag{5}$$

The same result holds for the lines AB_1, AC_1, AD_1 for which the ratios of the sines are the inverses of those in (5).

COROLLARY. *The isogonals of the points A', B', C', D', with respect to the triangles formed by the faces BCD, CDA, DAB, ABC of T coincide with the vertices of a tetrahedron which is at the same time inscribed and circumscribed to T.*

This proposition is mutually related to the first corollary of Section 1.

3. A particular case. If the points A', B', C', D' lie on a straight line Δ, they may be regarded as the vertices of a degenerate tetrahedron T', whose faces are the planes that join Δ to the vertices of T and each of T, T' is inscribed and circumscribed to the other. Thus,

If a straight line Δ cuts the planes of the faces BCD, CDA, DAB, ABC of a tetrahedron T at A', B', C', D', respectively, then the isogonals of the lines AA', BB', CC', DD', meet these planes in the vertices A_1, B_1, C_1, D_1 of a tetrahedron at the same time inscribed and circumscribed to T [2].

COROLLARY. *The isogonals of the points A', B', C', D', with respect to the triangles of the faces BCD, CDA, DAB, ABC, coincide with the vertices of a tetrahedron at the same time inscribed and circumscribed to T.*

References

1. Conderc et Ballicioni, Premier Livre du tetraedre, p. 26.
2. J. Hecquet, Mathesis, 1958, p. 112, question 3838.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

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CONCERNING SOME INEQUALITIES

ALBERT A. MULLIN, University of Illinois

This note is concerned with the use of concave function theory in deriving a particular set of inequalities.

DEFINITION 1. *A set of points S is said to be convex* if for $x \in S$, $y \in S$, and $0 \leqq \lambda \leqq 1$, all points of the form $(1-\lambda)x+\lambda y$ are elements of S.*

DEFINITION 2. *A function f is called concave if its domain is a convex set, its range is a subset of E^1, and for $0 \leqq \lambda \leqq 1$, $f[(1-\lambda)x+\lambda y] \geqq (1-\lambda)f(x)+\lambda f(y)$.*

LEMMA 1.1. *The logarithmic function is a concave function.*

Proof. This follows by observing that the second derivative of $\ln x$, which is $-1/x^2$, is negative for all $x>0$. Hence, each and every chord joining two distinct points of the logarithmic curve does not lie above the logarithmic curve.

LEMMA 1.2. *If $a>0$, $b>0$, $\alpha \geqq 0$, $\beta \geqq 0$, and $\alpha+\beta=1$, then $\alpha a+\beta b$ is a convex set.*

Proof. This follows directly from the definition of a convex set.

THEOREM 1. *If $a>0$, $b>0$, $\alpha \geqq 0$, $\beta \geqq 0$, and $\alpha+\beta=1$, then $\alpha \ln a+\beta \ln b \leqq \ln(\alpha a+\beta b)$.*

Proof. The theorem follows from Definition 2 and Lemmas 1.1, 1.2.

COROLLARY 1.1.

$$\frac{a^{2\alpha}b^{2\beta}}{\alpha a+\beta b} \leqq a^{\alpha}b^{\beta} \leqq \alpha a+\beta b.$$

Proof. The right-hand inequality follows directly from Theorem 1. The left-hand inequality follows by noticing that

$$a^{\alpha}b^{\beta} \equiv \frac{a^{2\alpha}b^{2\beta}}{a^{\alpha}b^{\beta}} \leqq \alpha a+\beta b,$$

or, since $\alpha a+\beta b>0$, then by cross-multiplying by $a^{\alpha}b^{\beta}$ and then dividing through by $\alpha a+\beta b$, we obtain the required inequality.

* H. G. Eggleston, Convexity, Cambridge, 1958.

COROLLARY 1.2.

$$\frac{2ab}{a+b} \leqq (ab)^{1/2} \leqq \frac{a+b}{2},$$

that is, the harmonic mean of two positive numbers does not exceed their geometric mean, which in turn does not exceed their arithmetic mean.

Proof. Put $\alpha=\beta=1/2$ in Corollary 1.1.

LEMMA 2.1. *If* $x>0$ *then* $1-(1/x)\leqq \ln x \leqq x-1$.

Proof. Consider the right-hand inequality. First, let

$$\lambda(x) \equiv \ln x - x + 1, \qquad x > 0.$$

Then $\lambda'(x)=(1/x)-1$, $\lambda''(x)=-1/x^2$. From this it follows that $x=1$ yields the only zero of λ' and for $x=1$, $\lambda''=-1$. Hence $x=1$ yields a maximum (and necessarily the only extrema) for λ and this proves the right-hand inequality.

Now consider the left-hand inequality. From the right-hand inequality with x replaced by $1/x$ it follows that $\ln(1/x)\leqq(1/x)-1$ and hence that $\ln x \geqq 1-(1/x)$. This yields the left-hand inequality.

LEMMA 2.2. *If* $a>0$, $b>0$, $\alpha\geqq 0$, $\beta\geqq 0$, *and* $\alpha+\beta=1$, *then* $\alpha \ln a+\beta \ln b \leqq \ln(\alpha a+\beta b)\leqq \alpha a+\beta b-1$.

Proof. The left-hand inequality is Theorem 1. The right-hand inequality results from an application of Lemma 2.1.

THEOREM 2.* *If* $a>0$, $b>0$, $\alpha\geqq 0$, $\beta\geqq 0$, *and* $\alpha+\beta=1$, *then*

$$\frac{\alpha}{a}[a+(a-1)^2]+\frac{\beta}{b}[b+(b-1)^2] \geqq \alpha a+\beta b-1+\frac{1}{\alpha a+\beta b} \geqq 1.$$

Proof. Consider the left-hand inequality first. The left-hand set of terms of this inequality may be put in the form $\alpha f(a)+\beta f(b)$, where

$$f(x) \equiv x-1+(1/x),\ x>0; \qquad f'(x)=1-(1/x^2); \qquad f''(x)=2/x^3.$$

Since $f''(x)>0$ for all $x>0$, each and every chord joining two distinct points of the curve $y=f(x)$ does not lie below the curve. Such a function is called a convex function. Hence $\alpha f(a)+\beta f(b)\geqq f(\alpha a+\beta b)$ and this proves the left-hand inequality.

The right-hand inequality is a result of $f(x)\equiv x-1+(1/x)$ having a minimum value of 1 (at $x=1$).

COROLLARY 2.1. *Let* $A\equiv(a+b)/2$ *and* $H\equiv(2ab)/(a+b)$ *denote the arithmetic and harmonic mean, respectively, of* a *and* b. *Then* $A+(1/H)\geqq 2$.

* The middle set of terms of the two inequalities appearing in Theorem 2 have been included at the suggestion of Professor Harry Levy, University of Illinois.

Proof. From Theorem 2 with $\alpha=\beta=1/2$ we obtain

$$\frac{1}{2}\left[a-1+\frac{1}{a}\right]+\frac{1}{2}\left[b-1+\frac{1}{b}\right]\geqq 1,$$

$$\frac{1}{2}[a+b]+\frac{1}{2}\left[\frac{1}{a}+\frac{1}{b}\right]\geqq 2,$$

from which the required inequality follows.

COROLLARY 2.2. *If* $\lambda>0$ *then* $\lambda+(1/\lambda)\geqq 2$.

Proof. Let $\lambda=a=b>0$ in Corollary 2.1.

THE CATENARY AND THE TRACTRIX

ROBERT C. YATES, College of William and Mary

Although the facts herein have been in the public domain for a long time, our purpose is to re-establish them in an elementary way—then set them on parade in a relatively complete and ordered line of march. Note that we carefully avoid rectangular equations (*per se*) which, at least for the tractrix, have an awesome appearance. Moreover, once a start is made, facts seem to tumble over each other in their demand for recognition.

1. The catenary. A flexible and inextensible chain weighing m pounds per foot hangs from two supports A and B. Equilibrium throughout is established if the three forces: T, the magnitude of the variable tangential force at a repre-

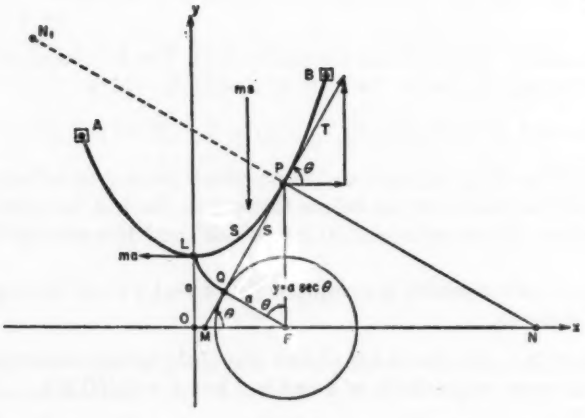

FIG. 1

sentative point P; ma, the constant* horizontal force at the lowest point L; and ms, the vertical weight of the chain length $s=\widehat{LP}$; have a resultant of zero magnitude. That is, if θ is the inclination of T:

$$T \sin\theta = ms, \qquad T\cos\theta = ma, \tag{1}$$

so that

$$s = a\tan\theta, \tag{2}$$

an *intrinsic* equation of the curve, called the catenary. Since the chain hangs steady, T is tangent to its locus at P and $\tan\theta$ is the slope of the curve.

In Figure 1, horizontal and vertical x, y axes are drawn with origin O distant a units below L. Let F be the foot of the ordinate of P and draw FQ perpendicular to the tangent line PT. Then, since $\measuredangle QFP=\theta$,

$$PQ = ks, \qquad FQ = ka, \qquad y = ka\cdot\sec\theta.$$

But for $\theta=0$, $y=a$, and thus $k=1$. Accordingly, the tangent to the catenary at any point is also tangent to the circle with center F and radius a.

From Equation (1), the tension is

$$T = ma\cdot\sec\theta = my, \tag{3}$$

a quantity equal to the weight of a length of the chain hanging vertically from P to F.

Since $y=a\cdot\sec\theta$ and generally $dx=(\cos\theta)ds$, then $ydx=(a\sec\theta)(\cos\theta)ds$ $=a\cdot ds$ and thus

$$\text{Area } (OLPF) = \int_0^x ydx = \int_0^s ads = as.$$

That is,

$$\text{Area } (OLPF) = 2\cdot\text{Area } (\Delta FQP). \tag{4}$$

Furthermore, since $ydx=ads$ or $\pi y^2dx=\pi ayds$, volume V_x and surface area $\sum_x$ of revolution about the x-axis have the special relation

$$2V_x = a\sum_x. \tag{5}$$

Let PN be the normal length from P to the x-axis. Then, if R is the radius of curvature at P, we have by definition from Equation (2):

$$R = \left|\frac{ds}{d\theta}\right| = a\sec^2\theta = \frac{y^2}{a} = PN. \tag{6}$$

The center of curvature, however, is at N_1 opposite N from P.

* This is evident if we imagine resupporting the chain with pegs at various points P. The shape of the chain does not change and thus the tension at L is constant in direction and magnitude.

2. The tractrix. In Figure 1, $PQ = s = \widehat{PL}$ and thus the locus of Q is an *involute* of the catenary, a curve called the tractrix. Its tangent length QF is con-

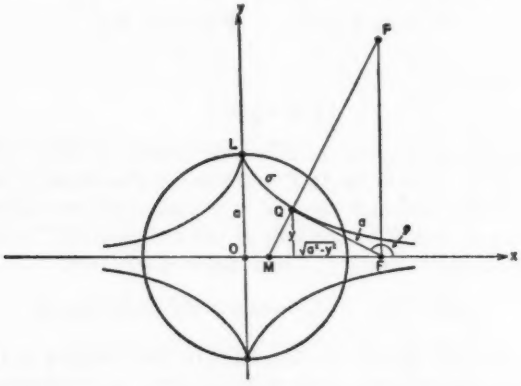

FIG. 2

stant and equal to a, and the curve can be pictured as the path of a toy wagon Q pulled along by a child F. It is quite obviously the orthogonal trajectory of circles of fixed radius a having centers on a line.

If ϕ is the inclination of the tangent, then the expression

$$\tan\phi = y' = \frac{y}{\pm\sqrt{a^2 - y^2}}, \qquad y = a,\ x = 0 \tag{7}$$

defines four branches as shown in Figure 2.

Interesting and useful properties of the tractrix are now established directly from this differential equation (7) which we write as $ydx = \pm\sqrt{a^2 - y^2}dy$. Note first, however, that if $\sigma = \widehat{LQ}$, $dy = (\sin\phi)d\sigma$ and, particularly here, $y = a\sin\phi$. The following measures of area A, radius of curvature ρ, volume V_x and surface area $\sum_x$ of revolution about the x-axis are immediate.

$$A = 2\int_{-\infty}^{\infty} ydx = 2\int_{-a}^{a} \sqrt{a^2 - y^2}\,dy = \pi a^2;^*$$

$$V_x = \pi\int_{-\infty}^{\infty} y^2 dx = 2\pi\int_0^a \sqrt{a^2 - y^2}\,(ydy) = \frac{2}{3}\pi a^3;$$

$$\sum_x = 2\pi\int_{-\infty}^{\infty} yd\sigma = 2\pi\int_0^a (a\sin\phi)(\csc\phi\,dy) = 2\pi\int_0^a ady = 2\pi a^2;$$

$$\rho = \frac{d\sigma}{d\phi} = |\,a\cot\phi\,| = |\,-a\tan\theta\,| = PQ.$$

* The form $\int_{-a}^{a}\sqrt{a^2-y^2}dy$ measures the half-area of the circle $x^2+y^2=a^2$ shown in Figure 2.

The last item is recognized as the involute-evolute relation of the catenary and tractrix. From it,

$$\sigma = |\, a \ln \sin \phi \,|\,, \tag{8}$$

an intrinsic equation of the tractrix.

The tractrix and its surface of revolution, called the *pseudosphere*, thus have much in common with the circle and sphere. This striking analogy appears even stronger if curvatures of the two surfaces are compared.

This curvature is determined as follows. A plane containing the normal to a surface at Q intersects the surface in a curve of curvature K. As the plane turns about the fixed normal, K may attain maximum and minimum values K_1 and K_2. Their *product* K_1K_2 is defined as the curvature of the surface at Q. These values K_1, K_2 occur in sections at right angles to each other.*

For a sphere all plane sections through a normal are great circles of radius a and curvature $1/a$. A sphere then has curvature $1/a^2$.

For the pseudosphere, the section of minimum curvature at Q is made by the plane through the axis of revolution; the maximum radius of curvature is QP. The minimum radius of curvature at Q is QM, formed by the plane perpendicular to the first. Their product $(QM)(QP) = -(FQ)^2 = -a^2$ is constant and negative since the radii are oppositely directed. Thus $K_1K_2 = -1/a^2$.

The plane, the sphere, and the pseudosphere are surfaces upon which we may display the parabolic geometry of Euclid, the elliptic geometry of Riemann, and the hyperbolic geometry of Lobatschewsky and Bolyai. These are characterized by the angle-sum $A+B+C$ of triangles formed by geodesics (lines of shortest distance):

$$A + B + C \left\{ \begin{matrix} < \\ = \\ > \end{matrix} \right\} 180^\circ \begin{cases} \text{hyperbolic} \\ \text{parabolic} \\ \text{elliptic.} \end{cases}$$

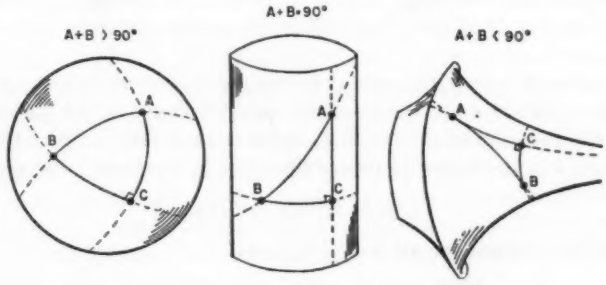

FIG. 3

* Along the *principal* directions of the surface.

A right circular cylinder may replace the plane. Figure 3 gives these surfaces showing triangles with $C=90°$. Their sides are each geodesics from point to point.

On the cylinder A and C are taken on an element, B and C on a circle. A and B lie on a *helix*.

On the sphere the sides are great circles.

On the pseudosphere A and C are on a tractrix (a meridian of the surface), B and C on a circle.

It appears that upon the sphere $A+B>90°$ and upon the pseudosphere $A+B<90°$. The fact that $A+B=90°$ on the cylinder is evident if we imagine the cylinder as a roller in a printing press. The image of ABC printed on plane paper is a triangle with straight sides. Moreover, on the cylinder

$$\widehat{AB}^2 = \widehat{BC}^2 + \overline{CA}^2.$$

As a final item, consider a pivot seated in a step. As the pivot turns, most wear occurs on the surface farthest from the axis of rotation. In time the seat

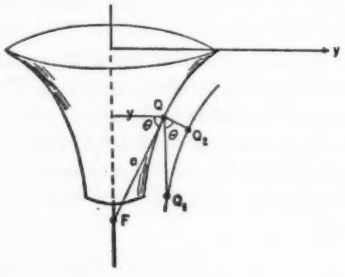

FIG. 4

and pivot become incompatible and wobble occurs. We seek the shape of a pivot such that the wear QQ_1, Figure 4, parallel to the axis of rotation is the same for all points Q of the pivot. Thus, as action and wear go on, the pivot will reseat itself.

The amount of wear QQ_2 normal to the section curve at $Q(x, y)$ is proportional to the work done by friction as the pivot turns. Let f be the coefficient of friction, p the pressure of the bearing, and n the number of revolutions per unit time. Then, if k is the factor of proportionality (a hardness constant),

$$QQ_2 = k(f)(p)(n)(2\pi y).$$

If QQ_1 is to be constant for all points Q, then

$$QQ_1 = (QQ_2)\sec\theta = (kfpn)(y\sec\theta) = \text{constant},$$

where θ is the angle between the normal and the axis. In short, $y\sec\theta=a$, a constant. Thus, if QF be drawn tangent to the curve, $QF=a$, a definitive

property of the tractrix. This is the form of the *Schiele* pivot mentioned in some books on Mechanics.

We end the account with a list of rectangular equations of the tractrix:

$$x = a \sinh^{-1} \sqrt{a^2y^{-2} - 1} - \sqrt{a^2 - y^2}$$

$$x = a \operatorname{sech}^{-1} \frac{y}{a} - \sqrt{a^2 - y^2}$$

$$x = \frac{a}{2} \ln \frac{a + \sqrt{a^2 - y^2}}{a - \sqrt{a^2 - y^2}} - \sqrt{a^2 - y^2}$$

$$x = t - a \tanh \frac{t}{a}, \qquad y = a \operatorname{sech} \frac{t}{a}.$$

They seem relatively useless.

MATHEMATICAL EDUCATIONAL NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
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GEOMETRY IN THE FIRST GRADE

NEWTON S. HAWLEY AND PATRICK SUPPES, Stanford University

In the spring of 1958 we spent two and a half months in an experiment which involved teaching geometrical notions and constructions to the entire class of first grade students at Stanford Elementary School (a public school in the Palo Alto Unified School District). After a few informal talks we began a systematic development of a modified version of Book I of Euclid's *Elements*.

Our approach was to stimulate reasoning among the pupils, although no formal proofs were attempted. The propositions from Euclid which were stressed were the constructions. In each case the construction (*e.g.*, bisecting a line segment) was presented as an open problem, and the students were encouraged to attempt solutions. As much as possible we forced the children to give the reasons for rejecting an incorrect solution. Generally speaking, our pedagogical procedure closely resembled that of Socrates' interrogation of the slave in Plato's dialogue *Meno*. It is worth noting that at no point did we rely on any knowledge of arithmetic. As a consequence our program is completely independent of the standard curriculum in elementary school mathematics.

Our main modification of Euclid was to use the compasses as rigid instruments to make direct comparisons of distances. We thereby trivialized Propositions 2 and 3 of Book I.

An encouraging aspect of the experiment was the unexpected ability of the students to assimilate the technical vocabulary. Printed worksheets were distributed every other day, and the students were asked to read aloud the written formulation of the problems. The technical terms introduced were then added to the regular reading vocabulary lists ordinarily presented in the first grade. Precision of expression was improved by daily drill and continual repetition of fundamentals. Each school day one of us met with the students for a period of from fifteen to thirty minutes.

It is our conclusion that the actual physical use by each pupil of straight edge and compass played an integral part in the progress of his understanding of geometrical constructions. This physical activity, which had direct mathematical meaning, seems to us somewhat superior to the conventional physical manipulation of objects used to illustrate arithmetic.

Certain problems arose due to disparities in the abilities of the students, but these disparities were no more striking than those encountered by the regular teacher in the standard reading program.

On the basis of the encouraging results of this experiment, we are continuing work along these lines, and are also exploring possibilities in other primary grades.

THE INDIANA SCHOOL AND COLLEGE COMMITTEE ON MATHEMATICS

G. N. Wollan, Purdue University

The current wave of interest in revision of the mathematics program in our schools has led in Indiana to the organization of a joint committee of school, college and university mathematics teachers and mathematicians whose purposes, very broadly stated, are (1) to stimulate and provide opportunities on a state-wide basis for critical analysis and discussion by mathematics teachers and by other interested persons of the various proposals for changing the mathematics program and (2) to develop a program of activities of various kinds on a state-wide basis designed to improve mathematics teaching. The initial impetus for the organization of this Committee was provided by college and university mathematicians acting individually and as members of the Indiana Section of the Mathematical Association of America. In particular, Professor Carl Kossack, Head of the Department of Mathematics and Statistics at Purdue University, took the initiative by inviting representatives of the mathematics departments of other Indiana colleges and universities to a meeting at Purdue in July, 1958. This led to the formation in September of the Indiana School and College Committee on Mathematics.

Having the support from the beginning of the college and university mathematicians of the State, the Committee has sought to enlist the support and active participation of elementary and high school mathematics teachers and administrators and the State Department of Public Instruction. Currently the Committee includes about twenty representatives of the mathematics depart-

ments of fifteen Indiana colleges and universities, about the same number of junior and senior high school mathematics teachers, and officially designated representatives of the following organizations: Indiana Council of Teachers of Mathematics, Indiana Secondary School Principals Association, Indiana County Superintendents Association, and the Indiana Association of Junior and Senior High School Principals. A representative of the Indiana State Department of Public Instruction has attended some of the Committee's meetings and expressed support of the Committee and its program. The Committee has the status of a standing committee of the Indiana Academy of Science and a committee of the Indiana Section of the Mathematical Association of America. However, it has no official governmental status and no authority with respect to the school curriculum. Hopes for its being effective rest upon the assumption that its activities are effective and that if it is sufficiently broadly representative, its conclusions will get official consideration.

Subcommittees have been organized to study the mathematics programs currently being followed in Indiana. There is a subcommittee for each of the following subdivisions: (1) Grades 1–6, (2) grades 7 and 8, (3) grades 9–11, (4) grade 12, (5) grades 13 and 14 (*i.e.* freshman and sophomore college years), (6) teacher training. Some of this work is underway. It is expected that a report of this study will be made public.

The committee has undertaken to promote the organization of regional meetings throughout the State for mathematics teachers. These meetings are intended to provide an opportunity for teachers to study and discuss mathematics and problems in the teaching of mathematics. Thus far, such meetings have been organized at eight colleges. Most of the meetings have been built around a talk on some mathematical topic followed by discussion. In one case, the local school system supports the venture as an in-service training program. At some of the colleges, these meetings are being held monthly. The speakers have so far been chosen almost exclusively from the college and university group. It seems to be agreed that it would be better to try to organize the meetings along the lines of a seminar with active participation of all members.

Among the N. S. F. summer institutes to be held this summer in Indiana, two are explicitly for Indiana teachers and will attempt to develop these teachers as active participants in the program of the Committee. At least partly as a result of the work of the Committee, National Science Foundation support has been obtained for Saturday and evening graduate credit courses in mathematics to be offered next year in conveniently located centers throughout the State.

SCHOOL MATHEMATICS STUDY GROUP

In order to keep the mathematical community informed of its plans and progress, the School Mathematics Study Group will issue from time to time a newsletter. Those who wish to receive this newsletter should so indicate on a postcard addressed to: School Mathematics Study Group, Drawer 2502A, Yale Station, New Haven, Conn.

PREPARATION OF HIGH SCHOOL TEACHERS OF SCIENCE AND MATHEMATICS

The Cooperative Committee on the Teaching of Science and Mathematics of the American Association for the Advancement of Science has recently prepared detailed recommendations on the Preparation of High School Teachers of Science and Mathematics.* The report was prepared by a subcommittee of the Cooperative Committee, with A. B. Garrett, Acting Chairman, Department of Chemistry, Ohio State University, as chairman. The Committee plans to refer the report to some sixteen scientific societies represented on the Committee and to revise the report in one year if suggestions from the societies appear to make this desirable. Phillip S. Jones is the representative of the Mathematical Association of America.

In the report are sections on the preparation of high school teachers of biology, chemistry, physics, physical sciences, mathematics, and general science. For the four year programs for science teachers, the requirements are summarized in the table below. This synthesis of recommendations also suggests multiple qualification in secondary school science; for example, a teacher prepared in biology could also qualify in general science by taking four additional units in physics and three additional units in earth science. The numbers represent semester hours.

	Common Foundation	High School Teacher of				
		Biology	Chemistry	Physics	Physical Science	General Science
Biology	6	24	2			4
Chemistry	8	4	20	5	10	
Physics	8			17	10	4
Earth Science	3				2	3
Mathematics	6		6	7	7	
Subtotal		28	28	29	29	11
Common Foundation		31	31	31	31	31+18†
Grand total		59	59	60	60	60

† 18 units upper division work selected from two or more of the four science fields (excluding mathematics) listed above.

Attention is called to the inclusion of mathematics as a part of the undergraduate program for all science teachers. An additional 2 to 6 hours in mathematics is suggested as a part of fifth year's work in all of the five programs for science teachers.

Part 5 of the report is given below in full.

† This report appears in full in *School Science and Mathematics*, Vol. LIX, No. 4, April 1959, page 281.

PART 5. SUGGESTED COURSES IN MATHEMATICS AND SCIENCE FOR HIGH SCHOOL TEACHERS OF MATHEMATICS
(Given in semester hours)

Suggested Courses	A	B	C	D	E	F	4-year Total		5th year		5-year Total	
							Math	Supporting Sciences	Math	Supporting Sciences	Math	Supporting Sciences
Teachers of Math only (Senior High School)	12	6	3	3	3	3	30	18	12	6	42	24
Teachers of Math only (Junior High School)	9	3	3	3	3	3	24	12	9	3	33	15
Teachers with Math minor (Junior or Senior High School)	9	3	3			3	18					

Notes on Part 5:

1. *Teaching Area*: There are three teaching areas distinguished in this report:
   (1) Mathematics in the senior high school.
   (2) Mathematics in the junior high school.
   (3) Mathematics combined with some other field such as science, history, or some other field at the junior or senior high level. Even though teaching assignments are not as desirable as those with only mathematics, such combinations must be recognized.
2. *Mathematics Courses:* On the undergraduate level minimum requirements in each section of mathematics are specified. On the graduate level only the total additional number of hours in mathematics is specified.
   A. *Analysis:* Trigonometry, plane analytic geometry, calculus (at least six semester hours required), solid analytic geometry, advanced calculus, differential equations, infinite series.
   B. *Applications:* Mechanics (statics and dynamics), mathematical physics, astronomy (mathematical), actuarial mathematics (finite differences, interpolation, numerical analysis), uses in behavioral science (theory of gases, linear programming, operations research).
   C. *Probability and Statistics:* Emphasizing probability and statistical inference.
   D. *Algebra:* Abstract algebra (fields, rings, groups, linear algebra, vector spaces), matrices, theory of numbers, theory of equations.
   E. *Geometry:* Metric and other geometries (projective, affine, inversive), non-Euclidean geometries, differential geometry, topology.
   F. *Foundations of Mathematics:* Theory of sets, mathematical or symbolic logic, postulates for geometry, postulates for algebra, postulates for arith-

metic, the real and complex number systems.

3. *Supporting Science Courses:* Each undergraduate requirement should include a course in physics. The other hours should be chosen from the following: chemistry, biology, astronomy, geology, and meteorology.

RATINGS OF COLLEGE MATHEMATICS COURSES BY APPLIED MATHEMATICIANS

R. W. HART AND WALTER H. WOOD, Kansas State College, Pittsburg, Kansas

The present emphasis on mathematics and the demand for mathematicians by industry leads college teachers to re-evaluate their course offerings in mathematics. Most college teachers know from experience what training is needed to prepare a student to teach elementary mathematics, but only a relatively few teachers have had enough practical experience in industry to know what courses are most valuable in this field.

In an effort to determine what college mathematics courses are most desirable to a graduate going into industry, a questionnaire was sent to twenty large concerns that hire mathematicians, asking them to rate various college mathematics courses according to their desirability for prospective employees. Twenty-nine head mathematicians of the following organizations answered the questionnaire: Air Research and Development Command, USAF; Armour Research Foundation; Boeing Aircraft Corporation—Wichita Division; Continental Oil Company; Department of the Navy; Firestone Tire and Rubber Company; General Electric Company; General Motors Corporation; International Business Machines Corporation; Lockheed Aircraft Corporation; Martin Company; Midwest Research Institute; Phillips Petroleum Company; United Aircraft Corporation, Research Department; Western Electric Corporation; Westinghouse Electric Corporation.

These will be recognized as some of the largest organizations in the United States engaged in such industries as: oil, aircraft, electricity, business machines, automobiles, guided missiles and research. They are a sample of those that are now doing the research that is so vital to our national progress. The usual college courses above the sophomore level were listed, and these head mathematicians rated them as "most desirable," "desirable" or "not needed." They are here listed in the decreasing order of desirability. The numbers are percentages of the replies from the twenty-nine mathematicians. Some did not rank all of the courses, hence there is not one hundred percent of replies on all of the subjects.

	Most Desirable	Desirable	Not Needed
Differential Equations	79	21	—
Applied Mechanics	72	14	10
Advanced Calculus	69	24	7
Matrix Theory	69	28	—
Mathematical Statistics	66	28	6
Theory of Probability	62	35	3
Functions of Complex Variables	62	31	7

Numerical Analysis	62	31	7
Higher Algebra	59	38	3
Vector Analysis	55	42	3
Theoretical Physics	49	39	12
Fourier Series	46	46	8
Theory of Equations	38	48	—
Functions of Real Variables	28	55	14
Boolean Algebra	21	52	14
Advanced Analytical Geometry	17	41	38
Statistical Quality Control	14	38	45
Modern Synthetic Geometry	14	17	55
Calculus of Variations	10	76	10
Introduction to Mathematical Thought	10	38	34
Projective Geometry	10	34	45
Integral Equations	7	76	14
Number Theory	7	38	41
Differential Geometry	3	55	34
Theory of Groups	3	52	34
Non-Euclidean Geometry	3	38	48
Topology	3	31	52
Higher Plane Curves	—	41	48

The following table lists those courses that were rated as "most desirable" by more than forty per cent of the mathematicians.

	%
Differential Equations	79
Applied Mechanics	72
Advanced Calculus	69
Matrix Theory	69
Mathematical Statistics	66
Theory of Probability	62
Functions of Complex Variables	62
Numerical Analysis	62
Higher Algebra	59
Vector Analysis	55
Theoretical Physics	49
Fourier Series	46

The courses that were marked as "not needed" by more than thirty percent of the mathematicians are given in the following table:

Modern Synthetic Geometry	55
Topology	52
Non-Euclidean Geometry	48
Higher Plane Curves	48
Projective Geometry	45
Statistical Quality Control	45
Number Theory	41
Advanced Analytical Geometry	38
Introduction to Mathematical Thought	34
Differential Geometry	34
Theory of Groups	34

The choice of the term "not needed" was an unfortunate one; because, as some of those who checked the questionnaire pointed out, it is difficult to say that any course in mathematics is not needed. The expression "least desirable" probably would have been better. However, the answers do give a good indication of what subjects are used in applied work.

Some valuable suggestions were made by those who participated in this project. One of the most frequent comments was that mathematics as taught in college tends to be too departmentalized. An applied mathematician needs to correlate his work with other fields. Others stated that if an employee has good training in basic mathematics and logical thinking, their companies could give him the special work needed in their particular occupations.

These ratings should be helpful to a mathematics major who intends to go into industry. They could serve as a guide to him and his advisors in the selection of subjects for study. Further study could be made on the content of courses. It would also be worthwhile to determine what courses in other departments are most desirable for applied mathematicians.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This Department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1371. *Proposed by G. C. Thompson, Security Mutual Life Ins. Co., Binghamton, N. Y.*

On p. 457 of Vol. I of Dickson's *History of the Theory of Numbers* appears the statement: "E. Barbier asked what is the 10^{1000}th digit written if the series of natural numbers be written down." What is this digit?

E 1372. *Proposed by J. R. Munkres, Princeton University*

Let $f(x)$ be continuous on the closed interval $[0, 1]$ and have a derivative on the open interval $(0, 1)$. Suppose $|xf'(x)-f(x)+f(0)| < x^2M$ for all x on $(0, 1)$, where M is fixed. Does $f'(0)$ exist?

E 1373. *Proposed by W. A. Veech, Dartmouth College*

Find the radius of convergence and the function $f(z)$ represented by

$$f(z) = \sum_{n=0}^{\infty} s_n z^n \quad \text{if} \quad s_n = \sum_{k=0}^{n} k\binom{n}{k}.$$

E 1374. *Proposed by Yonder Page, Kitchawan, N. Y.*

What is the minimum d such that the unit square may be covered with five sets each having diameter d?

E 1375. *Proposed by L. D. Goldstone, N. Y. State Public Works Lab., Albany, N. Y.*

Construct a triangle given A, m_a, t_a, that is, given a vertex angle and the median and angle bisector issued from this vertex.

SOLUTIONS

Real and Imaginary Parts of z^z

E 1341 [1958, 774]. *Proposed by A. A. Mullin, University of Illinois*

If $w=z^z$, $z=x+iy$, express Re (w) and Im (w) in terms of x and y.

Solution by David Zeitlin, Remington Rand Univac. Assuming $z\neq 0$, let $r=(x^2+y^2)^{1/2}$ and $\theta=\text{Arctan}\,(y/x)+2m\pi$, $m=0, \pm 1, \pm 2, \cdots$. Then

$$w = z^z = e^{z \log z} = e^{(x+iy)(\log r+i\theta)} = e^{x \log r-\theta y+i(y \log r+\theta x)}.$$

Thus

$$\text{Re } (w) = e^{x \log r-\theta y} \cos (y \log r + \theta x),$$
$$\text{Im } (w) = e^{x \log r-\theta y} \sin (y \log r + \theta x).$$

Also solved by R. G. Albert, Norman Anning, M. T. Austin, Robert Bart, A. P. Boblétt, D. A. Breault, J. L. Brown, Jr., R. F. Brown and Joel Levy (jointly), P. L. Chessin, R. J. Cormier, E. S. Eby, Edwin Ellis, D. A. Freedman, Michael Goldberg, L. D. Goldstone, A. B. Harper, Jr., J. R. Holdsworth, A. M. Krall, Morris Morduchow, D. C. B. Marsh, Beckham Martin, D. L. Muench, C. S. Ogilvy, Sister Barbara Ann, W. A. Veech, and Dale Woods. Late solutions by A. E. Danese, Joseph Hammer, J. D. E. Konhauser, P. K. Maloof, J. B. Muskat, J. T. Parent, and J. W. Young.

The problem is stated in Ahlfors, *Complex Analysis* (1953), p. 49, and solved in Loney, *Plane Trigonometry*, Part II (1896), p. 101.

For Positive x and y, $x^y+y^x>1$

E 1342 [1958, 774]. *Proposed by D. J. Newman, AVCO Research Division, Wilmington, Mass.*

If x and y are positive, prove that $x^y+y^x>1$.

Solution by Viktors Linis, University of Ottawa. The inequality is trivially true if either x or $y\geqq 1$. Let $0<x, y<1$, and put $y=kx$. Because of the symmetry we consider only $0<k\leqq 1$. Now

$$f(x) = x^{kx} + (kx)^x = (x^x)^k + k^x x^x \geqq a^k + ka,$$

where $a=\exp(-1/e)=\min(x^x)$ and $k^x \geqq k$. But $F(k)=a^k+ka$ has a unique minimum at $k_0=1-e<0$ and is increasing for $k>k_0$. Since $F(0)=1$, $F(1)=2a>1$, $f(x)>1$.

Also solved by Robert Bart, R. F. Brown and Joel Levy (jointly), E. L. Ellis, Michael Goldberg, Martin Harrow, Fritz Herzog, C. S. Ogilvy, Dale Woods, and the proposer. Late solutions by J. W. Baldwin, J. H. Bramble, D. A. Breault, G. B. Charlesworth, P. G. Hodge, Jr., L. M. Lewandowski, G. H. Meisters and J. E. Sheats (jointly), W. S. Reid, D. I. Sommerville, and an anonymous solver.

A Geometrical Probability Problem

E 1343 [1958, 774]. *Proposed by W. F. Cheney, University of Hartford, Connecticut*

Through an arbitrarily selected point within a plane triangle, a straight line is drawn which bisects the area of the triangle. What is the probability of more than one solution for a given point?

Editorial Note. This is essentially E 1228 [1957, 198]. The answer is

$$p = (3 \log 2 - 2)/4 = 0.01986 \text{ approx.}$$

Solved by Michael Goldberg, L. D. Goldstone, Viktors Linis, D. C. B. Marsh, and Helen M. Marston. Late solution by John Burr.

Convergent Subsequences of Quotients

E 1344 [1958, 774]. *Proposed by A. J. Goldman, National Bureau of Standards, Washington, D. C.*

Let $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \cdots, \{x_n^{(m)}\}$ be m sequences of nonzero real numbers. Prove that there exists an integer i $(1 \leqq i \leqq m)$ and an ascending sequence $\{n_r\}$ of positive integers $(r=1, 2, \cdots)$ such that each of the sequences

$$\{x_{n_r}^{(i)}/x_{n_r}^{(1)}\}, \{x_{n_r}^{(i)}/x_{n_r}^{(2)}\}, \cdots, \{x_{n_r}^{(i)}/x_{n_r}^{(m)}\}$$

is convergent.

Solution by C. H. Cunkle, Cornell Aeronautical Laboratory, Inc., Buffalo, N. Y. Take x_n such that $|x_n| = \min |x_n^{(j)}|$ for $1 \leqq j \leqq m$. Then for some i $(1 \leqq i \leqq m)$, the sequence $\{x_n\}$ must have a subsequence in common with $\{x_n^{(i)}\}$. Call this subsequence $\{x_{n_s}^{(i)}\}$ and let $y_{n_s}^{(j)} = x_{n_s}^{(i)}/x_{n_s}^{(j)}$. Since $|y_{n_s}^{(j)}| \leqq 1$, the sequence $\{y_{n_s}^{(j)}\}$ is bounded for each j, and every subsequence thereof contains a convergent subsequence. Let $\{y_{n_{s_1}}^{(1)}\}$ be a convergent subsequence of $\{y_{n_s}^{(1)}\}$, and for $2 \leqq j \leqq m$, let $\{y_{n_{s_j}}^{(j)}\}$ be a convergent subsequence of $\{y_{n_{s_{j-1}}}^{(j)}\}$. Then $\{y_{n_{s_m}}^{(j)}\}$ is convergent for each j, and $\{n_{s_m}\}$ may be taken as the desired sequence $\{n_r\}$.

The property fails to be true for an infinite number of sequences $\{x_n^{(j)}\}$ even though each may be convergent itself. For example, let $x_n^{(j)} = j^{-n}$. One sees also that the nonzero hypothesis is necessary by letting $m=2$, $x_n^{(1)} = (1/n)\sin(n\pi/2)$, $x_n^{(2)} = (1/n)\cos(n\pi/2)$.

Also solved by R. F. Brown and Joel Levy (jointly), D. C. B. Marsh, W. A. Veech, and the proposer. Late solution by John Burr.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4851. *Proposed by Z. A. Melzak, McGill University, Montreal*

Let X be the set of all rational numbers of the open interval (0, 1). Prove that for some enumeration $\{x_n\}$ of X, the series

$$S = \sum_{n=1}^{\infty} \prod_{j=1}^{n} x_j$$

diverges.

4852. *Proposed by P. T. Bateman, University of Illinois*

Give an example of a ring with a finite number of elements which can not be isomorphic to a residue-class ring of the ring of integers of an algebraic number field (of finite degree over the rationals).

4853. *Proposed by D. J. Newman, AVCO Research and Development, Wilmington, Mass.*

The dimension of a Banach space can be defined, *e.g.*, in the following two ways: (1) the smallest number (cardinal) of complex numbers necessary to specify a vector; (2) the smallest number of vectors such that linear combinations of them are dense. Are these two definitions equivalent?

4854. *Proposed by Ky Fan, Oak Ridge National Laboratory*

Let $\{\alpha_i\}$, $\{\beta_i\}$ $(1 \leqq i \leqq n)$ and γ be given complex numbers, each of absolute value 1. Prove: There exist two unitary matrices A, B of order n with the preassigned eigenvalues $\{\alpha_i\}$, $\{\beta_i\}$ respectively, such that γ is an eigenvalue of AB, if and only if, in the complex plane, the convex hull of the n points $\{\alpha_i\}$ and the convex hull of the n points $\{\gamma\bar{\beta}_i\}$ have a point in common.

4855. *Proposed by M. S. Klamkin, AVCO Research and Development, Wilmington, Mass.*

The angle of intersection between two spheres is constant for all points on any curve of intersection. Are there other surfaces with this property?

4856. *Proposed by L. A. Rubel, Institute for Advanced Study*

The growth function $h(\theta)$ of an entire function $f(z)$ of exponential type is defined by $h(\theta)=\limsup_{r\to\infty} r^{-1}\log|f(re^{i\theta})|$. It is well known that $f(z)\equiv f_1(z)f_2(z)$ need not imply that $h(\theta)\equiv h_1(\theta)+h_2(\theta)$. Suppose, however, that $h_1(\theta)\equiv h_2(\theta)$. Must $h(\theta)\equiv h_1(\theta)+h_2(\theta)$ in this case?

SOLUTIONS

A Theorem of Baire

4804 [1958, 633]. *Proposed by I. S. Gál, Cornell University.*

A theorem of Baire states that every lower semi-continuous function on a metric space is the pointwise limit of an increasing sequence of continuous functions. Show by an example that the theorem cannot be extended to arbitrary uniform spaces.

Solution by the proposer. Let S be a noncountable set and let $X=\{\infty\}\cup S$. Define a topology on X by specifying its closed sets: $C\subseteq X$ is closed if $\infty\in C$ or if it is countable. Under this topology X becomes a Hausdorff space. We show that X is completely normal: Let A and B be separated sets in X. If $\infty\notin A\cup B$ then A and B are open and so $O_A=A$ and $O_B=B$ are disjoint open neighborhoods of A and of B, respectively. If $\infty\in A$ and B is separated from A, then B must be closed. Indeed $\overline{B}=\cap[C: B\subseteq C]\subseteq\{\infty\}\cup B$ and so by $\overline{B}\subseteq cA$ we have $\overline{B}=B$. Hence A and B can be separated by the disjoint open neighborhoods $O_A=cB$ and $O_B=B$. Since the space is completely normal and separated, it is uniformizable.

Every real-valued function f on X is continuous at each $x\in S$. We prove that a function f is lower semi-continuous at ∞ if and only if there is an open set O_∞ containing ∞ such that $f(x)\geqq f(\infty)$ for every $x\in O_\infty$. In fact, if f is lower semi-continuous at ∞ then for every $\epsilon=1/n$, $(n=1, 2, \cdots)$, there is an open neighborhood $O_\infty^{(n)}$ of ∞ such that $f(x)>f(\infty)-1/n$ for every $x\in O_\infty^{(n)}$. Hence $f(x)\geqq f(\infty)$ for every $x\in O_\infty=\cap O_\infty^{(n)}$ and O_∞ is open because its complement is countable.

It follows that noncontinuous, lower semi-continuous functions exist. Also we see that f is continuous if and only if $f(x)=f(\infty)$ for every point x of an open neighborhood O_∞ of ∞. Hence if $f_1, f_2, \cdots$ is a sequence of continuous functions on X, then there exist open sets $O_\infty^{(n)}$ such that $f_n(x)=f_n(\infty)$ for every $x\in O_\infty^{(n)}$. Therefore $f_n(x)=f_n(\infty)$ for every $n=1, 2, \cdots$ and for every $x\in O_\infty=\cap O_\infty^{(n)}$. Now let the sequence be pointwise convergent to a limit function f. Then $f_n(\infty)\to f(\infty)$ and so $f_n(x)=f_n(\infty)\to f(\infty)$ for every $x\in O_\infty$. Hence $f(x)=f(\infty)$ for every $x\in O_\infty$ and so the pointwise limit of continuous functions is continuous.

A Double Summation

4805 [1958, 633]. *Proposed by R. R. Goldberg, Pittsburgh, Pa.*

The following relation is found to be true for $n=1, \cdots, 6$:

$$n^2 \sum_{k=0}^{n} \frac{(-1)^{n+k}}{(n-k)!} \frac{(n+k-1)!}{k!k!} \sum_{j=1}^{n+k-1} \frac{1}{j} = 1.$$

Is it true for all positive integers n?

Solution by Chih-yi Wang, University of Minnesota. The answer is in the affirmative. By writing $n^2=(n+k)(n-k)+k^2$, the left member may be transformed as follows:

$$\begin{aligned} & n^2 \sum_{k=0}^{n} \frac{(-1)^{n+k}}{(n-k)!} \frac{(n+k-1)!}{k!k!} \sum_{j=1}^{n+k-1} \frac{1}{j} \\ &= \sum_{k=0}^{n-1} \frac{(-1)^{n+k}}{(n-k-1)!} \frac{(n+k)!}{k!k!} \sum_{j=1}^{n+k-1} \frac{1}{j} + \sum_{k=1}^{n} \frac{(-1)^{n+k}}{(n-k)!} \frac{(n+k-1)!}{(k-1)!(k-1)!} \sum_{j=1}^{n+k-1} \frac{1}{j} \\ &= \sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(n-k-1)!} \frac{(n+k-1)!}{k!k!} = (-1)^{n+1} \sum_{k=0}^{n-1} (-1)^k \binom{n-1+k}{n-1} \binom{n-1}{k} \\ &= (-1)^{n+1}(-1)^{n-1}(1)^{n-1} = 1, \end{aligned}$$

where we have used the known formula

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{\alpha+\beta k}{n} = (-1)^n \beta^n, \qquad n \geqq 0.$$

See, for example, Gould, *Some Generalizations of Vandermonde's Convolution*, this MONTHLY, vol. 63, 1956, p. 85.

Also solved by Leonard Carlitz, Karl Goldberg, Y. L. Luke, E. Makai, Immanuel Marx D. J. Newman, F. D. Parker, R. C. Read, James Singer, and Franklin C. Smith.

Location of Zeros

4806 [1958, 633]. *Proposed by D. J. Newman, AVCO Research and Development, Wilmington, Mass.*

Let $0=a_0<a_1<a_2<\cdots$ be integers. Prove that $\sum_{n=0}^{\infty} z^{a_n}$ has no zeros in $|z|<(\sqrt{5}-1)/2$. Also, this is the best possible constant.

Solution by Leonard Carlitz, Duke University. Put $\alpha=(\sqrt{5}-1)/2$. So that $\alpha^2+\alpha-1=0$.

1. If $a_1>1$, then for $|z|<\alpha$, we have

$$\left| \sum_{n=1}^{\infty} z^{a_n} \right| < \left| \sum_{n=2}^{\infty} \alpha^n \right| = \frac{\alpha^2}{1-\alpha} = 1.$$

Hence $\sum_{n=0}^{\infty} z^{a_n} \neq 0$.

2. If $a_1=1$, consider

$$(1-z) \sum_{n=0}^{\infty} z^{a_n} = 1 - z^{b_1} + z^{b_2} - z^{b_3} + \cdots,$$

where $2 \leqq b_1 < b_2 < \cdots$. The proof is now completed exactly as above.

3. For the function $f(z) = 1+z+z^3+z^5+\cdots$, $(|z|<1)$, we have

$$f(z) = 1 + \frac{z}{1-z^2} = \frac{1+z-z^2}{1-z^2}.$$

Consequently $f(-\alpha) = 0$.

Also solved by J. B. Kelly, J. C. C. Nitsche, and the proposer.

A Diophantine Equation

4807 [1958, 633]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Determine the integral solutions of the equation

$$(m-1)x^2 + my^2 = (m+1)z^2. \tag{1}$$

Solution by R. Venkatachalam Iyer, Trivandrum, India. For certain values of m, but not for all, equation (1) has solutions. When, however, there exists one set of integers, α, β, γ, such that

$$(m-1)\alpha^2 + m\beta^2 = (m+1)\gamma^2, \tag{2}$$

then the general solution, for that value of m, can be obtained as follows. Put $x=\alpha$, $y=\beta+p\xi$, $z=\gamma+q\xi$ in (1) and solve for ξ. The result is

$$\xi = \frac{2m\beta p - 2(m+1)\gamma q}{(m+1)q^2 - mp^2}.$$

Hence, x, y, z may be chosen as

$$\begin{aligned} \pm x &= \alpha\{-mp^2 + (m+1)q^2\} \\ \pm y &= m\beta p^2 - 2(m+1)\gamma pq + (m+1)\beta q^2 \\ \pm z &= m\gamma p^2 - 2m\beta pq + (m+1)\gamma q^2, \end{aligned} \tag{3}$$

with p, q arbitrary, or any integers proportional to (3).

For small values of m, for which solutions exist, it is possible to determine α, β, γ by trial. In particular, if k is any integer we may take $m=k^2+1$, $\alpha=1$, $\beta=k$, $\gamma=k$; or $m=2k^2$, $\alpha=k$, $\beta=1$, $\gamma=k$.

Also solved by W. J. Blundon and by the proposer.

Editorial Note. The proposer puts (1) in the form $m=(x^2+z^2)/(x^2+y^2-z^2)$ from which solutions can be obtained with $m=x^2+z^2$ provided $x^2+y^2-z^2=1$. Solutions of the latter equation are known: Let a, b, c, d be integers such that $ac-bd=\pm 1$ and take

$$x = ac+bd, \qquad y = ad-bc, \qquad z = ad+bc. \tag{4}$$

With these values, $m=(ac+bd)^2+(ad+bc)^2$. Then for each m, the corresponding x, y, z give rise to an infinite set of solutions as indicated in (3) above.

To determine whether solutions exist for a proposed value of m we may employ a theorem of Legendre (*Théorie des nombres*, 1830, p. 47). *If each of the coefficients a, b, c is different from zero and has no square factor, if they are not all of the same sign and no two have a common factor, then the necessary and sufficient condition that solutions of $ax^2+by^2+cz^2=0$ exist is that the numbers $-bc$,*

$-ca$, $-ab$ shall be quadratic residues respectively of a, b, c. For $m>1$ the equation (1) can always easily be reduced to Legendre's form and the determination made.

To show that, for a given m, the general solution of (1) is given by (3), multiply (1) through by α^2 and replace $(m-1)\alpha^2$ by its value from (2). The result is

$$(m+1)(\gamma^2x^2 - \alpha^2z^2) = m(\beta^2x^2 - \alpha^2y^2).$$

Let x, y, z be any integers satisfying (1) and let p/q, $(p, q)=1$, be

$$(\beta x + \alpha y)/(\gamma x + \alpha z)$$

reduced to lowest terms. We now have

$$(m+1)q(\gamma x - \alpha z) = mp(\beta x - \alpha y), \qquad p(\gamma x + \alpha z) = q(\beta x + \alpha y),$$

from which the ratios of x, y, z are found to be as given in (3). Conversely, the values given in (3) satisfy (1) by virtue of (2).

By Legendre's theorem it is easily shown that m must be a sum of two squares. One important question remains unanswered: Is the proposer's solution (4) complete in the sense that for every m for which (1) has solutions, one of these solutions satisfies $x^2+z^2=m$ and $x^2+y^2-z^2=1$? No counterexample has been brought to light.

Bernoulli Numbers—A Finite Summation

4808 [1958, 633]. *Proposed by J. M. Gandhi, Jain Engineering College, Panchkoola, India*

Consider the expression $S_2(n, i) = \sum a_1^2 \cdots a_i^2$, where the sum is taken over all possible integral choices of the a's such that $1 \leqq a_1 \leqq \cdots \leqq a_i \leqq n$. Prove that

$$2(2^{2n}-1)B_{2n} = (-1)^{n-1}\{(n-1)!^2 - (n-2)!^2S_2(n-2, 1) + (n-3)!^2S_2(n-3, 2) - \cdots (-1)^n 1!^2 S_2(1, n)\},$$

where the B_{2n} are the well-known Bernoulli numbers.

Solution by R. C. Read, University College of the West Indies, Jamaica. Denote the right-hand side of the equation by X_{2n}, so that

$$X_{2n} = \sum_{r=1}^{n-1} (-1)^r r!^2 S_2(r, n-r-1).$$

$S_2(r, i)$ is the sum of the products i at a time, with repetitions allowed, of the squares of the numbers from 1 to r. Hence it is the coefficient of x^i in the expansion of

$$1/(1-x)(1-2^2x)(1-3^2x)\cdots(1-r^2x).$$

Consequently $S_2(r, n-r-1)$ is the coefficient of x^{n-r-1} in

$$\prod_{s=1}^{r}(1-s^2x)^{-1},$$

and $r!^2S_2(r, n-r-1)$ is thus the coefficient of x^{n-1} in

$$\prod_{s=1}^{r}\frac{s^2x}{1-s^2x}.$$

Hence X_{2n} equals the coefficient of x^{2n-2} in

$$\sum_{r=1}^{n-1}(-1)^r\prod_{s=1}^{r}\frac{s^2x^2}{1-s^2x^2}.$$

Resolving into partial fractions in the usual way we find that

$$\prod_{s=1}^{r}\frac{s^2x^2}{1-s^2x^2}=(-1)^r+\sum_{\alpha=1}^{r}\frac{A_\alpha^{(r)}}{1-\alpha^2x^2},$$

where $A_\alpha^{(r)}=2(-1)^{r-\alpha}r!^2/(r-\alpha)!(r+\alpha)!$. Hence X_{2n} equals the coefficient of x^{2n-2} in

$$\sum_{r=1}^{n-1}(-1)^r+\sum_{r=1}^{n-1}\sum_{\alpha=1}^{r}\frac{2(-1)^\alpha r!^2}{(r-\alpha)!(r+\alpha)!}\frac{1}{1-\alpha^2x^2}. \tag{1}$$

Consider the expression obtained from (1) by replacing x^m by $x^m/m!$. It is

$$\sum_{r=1}^{n}(-1)^r+\sum_{r=1}^{n-1}\sum_{\alpha=1}^{r}\frac{2(-1)^\alpha r!^2}{(r-\alpha)!(r+\alpha)!}\cosh\alpha x, \tag{2}$$

since $(1-\alpha^2x^2)^{-1}$ is replaced by $\cosh\alpha x$. (2) may now be written

$$\begin{aligned}\sum_{r=1}^{n-1}\frac{r!^2}{(2r)!}\left\{\sum_{\alpha=1}^{r}(-1)^\alpha\binom{2r}{r-\alpha}(e^{\alpha x}+e^{-\alpha x})+(-1)^r\binom{2r}{r}\right\}&\\ =\sum_{r=1}^{n-1}\frac{r!^2}{(2r)!}(-1)^r(e^{x/2}-e^{-x/2})^{2r}&\\ =\sum_{r=1}^{n-1}\frac{r!^2}{(2r)!}(-1)^r(2\sinh\tfrac{1}{2}x)^{2r}.&\end{aligned} \tag{3}$$

Hence, for $n>1$, X_{2n} is $(2n-2)!$ times the coefficient of x^{2n-2} in

$$\sum_{r=0}^{\infty}\frac{r!^2}{(2r)!}(-1)^r(2\sinh\tfrac{1}{2}x)^{2r},$$

since the expression differs from (3) only in the constant term and in terms of higher degree than $2n-2$.

Now it can easily be shown that

$$\sum_{r=0}^{\infty}(-1)^r\frac{r!^2}{(2r!)}(2z)^{2r}=\frac{1}{1+z^2}-\frac{z\sinh^{-1}z}{(1+z^2)^{3/2}},$$

and hence X_{2n} is $(2n-2)!$ times the coefficient of x^{2n-2} in $\operatorname{sech}^2\frac{1}{2}x-\frac{1}{2}x\operatorname{sech}\frac{1}{2}x\tanh\frac{1}{2}x=(d^2/dx^2)\{x\tanh\frac{1}{2}x\}$. But it is well known that

$$x\tanh\tfrac{1}{2}x=\sum_{n=0}^{\infty}\frac{2(2^{2n}-1)}{(2n)!}B_{2n}x^{2n},$$

so that

$$\frac{d^2}{dx^2}\{x \tanh \tfrac{1}{2}x\} = \sum_{n=1}^{\infty} \frac{2(2^{2n}-1)}{(2n-2)!} B_{2n}x^{2n-2}$$

whence $X_{2n}=2(2^{2n}-1)B_{2n}$.

Also solved by Leonard Carlitz.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

The Gyroscope; Theory and Applications. By J. B. Scarborough. Interscience, New York, 1958. 257 pp. $6.50.

This book gives a practical, unsophisticated and careful mathematical treatment of the theory and applications of the gyroscope. Two thirds of the book is devoted to a detailed engineering analysis of devices in which the gyroscopic principle is used and of situations in which gyroscopic action has a nonnegligible role. The topics covered include gyroscopic effects in car wheels, hoops, grinding mills, projectiles, propeller-type airplanes, the gyroscopic compass, the Draper hermetic integrating gyro, the Sperry gyrotron vibratory gyroscope, gyroscopic stabilizers, and astronomy. An appendix treats Schuler tuning, a technique devised to mitigate the effect of vehicle motion on a gyroscopic instrument.

The book should be usable from cover to cover by anyone who at some time has had a year of calculus and a course in engineering mechanics. Any deficiencies in his early training are supplied in the first third of the book by an eighteen-page exposition of vector algebra, a sixteen-page summary of the elements of mechanics of rigid bodies, and forty-six pages on gyroscopics in general. The reader with more background would do well to read chapters four and five of Goldstein's *Classical Mechanics* and chapter four of Sommerfeld's *Mechanics* before becoming enmeshed with the manipulative details of Scarborough's exposition.

This book has no exercises for students, but going through his many worked-out examples will serve the same purpose. I feel that the intended readers are engineers and not mathematicians or physicists. The book shows Scarborough's care and attention to detail, well known to us from his now justly famous *Numerical Mathematical Analysis.*

NATHAN GRIER PARKE, III
Parke Mathematical Labs., Inc.

Elements of Plane Trigonometry. By H. Sharp, Jr. Prentice-Hall, Englewood Cliffs, N. J., 1958. ix+274 pp. $4.95.

The first paragraph of the preface states: "The purpose of this textbook is to present trigonometry in the language and spirit of modern mathematics. The vocabulary of elementary mathematical analysis is used exclusively throughout."

The book begins with a brief introductory chapter in which the student is told that trigonometry is concerned with periodic phenomena as well as the mensuration of triangles. Next, careful definitions of *relation* and *function* are given together with enough examples to cement understanding. (Indeed, these definitions are stressed throughout the text.) Then the trigonometric functions are defined, and their elementary properties are thoroughly investigated. The main topics considered at this point are: Values at special angles, periodicity and graphs, trigonometric identities and equations, multiple angle formulas, and the inverses of trigonometric functions. (Before the latter is introduced, a careful discussion of inverse relation and inverse functions is given.) The main body of the text closes with applications of the above to solving triangles, periodic phenomena, and complex numbers and De Moivre's theorem. All this is accomplished in 168 pages. The remainder of the text consists of three appendices (scientific notation, numerical computation, and logarithms), sets of tables, an index, and answers to exercises.

The author manages to carry out his program without neglecting the standard topics, without being too brief for students at this level, and without turning a text into a treatise. (Evidently the author feels that a text should supplement rather than supplant competent instruction.) The author has done an exceedingly good job of placing virtually all of the topics normally covered in a standard trigonometry course in proper mathematical perspective.

This excellent book would be even better if it did not contain a number of minor annoyances. In some places the author uses semisymbolic jargon instead of a clear, understandable English sentence (*cf.*, *e.g.*, Examples 2 and 3, pp. 80–81).

MELVIN HENRIKSEN
Purdue University

The Tree of Mathematics. Managing Editor, Glenn James. The Digest Press, Pacoima, California, 1957. 403+xvii pages. $6.00.

This book is the joint work of some twenty contributors who have written brief expository chapters on the main branches of mathematics, beginning with high school algebra and extending into the currently active fields of the present day. These chapters appeared previously as a series of articles in the Mathematics Magazine, of which Dr. James is the managing editor.

The first seven chapters comprise a rapid account of elementary mathematics through the fundamental ideas of the calculus, and each of these chapters

concludes with a short list of exercises to test the reader's understanding. The material in these introductory chapters forms a minimum prerequisite for the comprehension of the succeeding expositions of advanced topics.

The topics covered include Differential Equations; Series; Theory of Numbers; Metric Differential, Non-Euclidean and Projective Geometry; Topology; Real and Complex Variable; Abstract Algebra; General Analysis; Integral Equations; Calculus of Variations; Matrices, Determinants, and Systems of Equations; Probability; Dynamic Programming; and the Theory of Games.

The authors have, in the limited space at their disposal, made their contributions meaningful and understandable. A reader with the prerequisite background and some natural mathematical ability should be able to grasp the central ideas of the topics presented.

The book contains an unfortunate number of misprints, some of which will confuse the reader who is seeing the material for the first time. Although the editor suggests that the book can be used for a survey course in high school or college, the reviewer believes that it is better suited for individual study by teachers and students of mathematics and related subjects who wish to extend their mathematical horizons. *The Tree of Mathematics* is a valuable contribution to mathematical expository literature.

R. A. Beaumont
University of Washington

Introductory Formal Logic of Mathematics. By P. H. Nidditch. University Tutorial Press, London, 1957. vii+188 pp. 12s 6d.

In a spirit of tolerance, we shall overlook the author's italicized comment "In the whole structure of mathematics there is not a single valid proof in the logical sense," as well as his generous spirit which he evidences in the remark, " . . . a part of the business of every mathematician is to give proofs; we show him how he can and ought to conduct his business." (p. v.) In a sense what Nidditch says is obvious. He considers a proof "logically valid" if *every* step taken is written down and justified by reference to logical rules, definitions, or preassigned descriptions. But this demand is also in many cases quite trivial, for there is hardly time to do this for every theorem. It becomes important when there is doubt that this construction can actually be effectuated either because of the mathematician's ability or the absence of some link in the chain. Quite obviously, however, Nidditch is not himself completely rigorous for he at times does skip steps and even assumes the consistency both of mathematics and of the rules of inference.

Be that as it may, the more direct aim of the book is to provide an introduction to formal logic using the methods and notation of the Polish logicians which are not common in the United States. This is done in a series of chapters presenting the "languages" of the propositional calculus, the quantificational calculus, the formula calculus, the calculus of set theory, elementary and complex. Each "language" is presented by listing its "alphabet" (symbols for terms, oper-

ations, and relations), the definitions (usually recursive) of its expressions and terms, the definitions of its formula, and its rules of inference. Careful explanations are given in very succinct form. In sum, for the more sophisticated formalist Nidditch accomplished his avowed aim quite respectably.

The mathematician interested in Boolean algebras will find on page 156 "five formulae" which "serve jointly to define any Boolean algebra of sets." Unfortunately he will not recognize them. This comment serves to point up a major criticism of the book so far as American logicians (for the most part) and mathematicians are concerned. It is not clear to the reviewer why, for example, Jb_1b_2 is better symbolically than $(b_1 \, v \, b_2) \cdot \sim (b_1b_2)$, or why $Eb_1b_2 = KCb_1b_2Cb_2b_1$ is to be preferred to $(b_1 \equiv b_2) \equiv (b_1 \subset b_2) \cdot (b_2 \subset b_1)$.

Indeed, the use of the operators E, K, C, *etc.* is blurred when symbols are used which too closely identify them with the term variables. In this connection it is interesting to note that the author apparently felt "English translations" of many of the formulae would help the student. These are provided in an appendix.

On page 63 Nidditch says that the requirement that all definitions and reasonings in mathematics be constructive leads the Intuitionists to eliminate "trex" (law of excluded middle). It is perhaps better to say that the Intuitionists *restrict* the use of this principle.

On page 65, Nidditch refers to the propositional, formula, and quantificational calculi as "pure logic" and the subsequent "languages" as mathematics. This is interesting, but points up what I believe is a confusion in Nidditch's point of view. As "languages" or "calculi" or better "algebras," all the formal systems are on a par. The only basis for calling L_{pc}, L_{fc} and L_{qc}, "pure logic" is in the interpretation one gives them. As abstract algebras (and I use this deliberately), they differ only in their definitions, formula, or alphabet.

Finally, it would help to know why, except for the fact that it is in fashion, these are called "languages" and not algebras.

Those interested in a more general introduction to formal logic using the Polish symbolism will find A. N. Prior's book useful. Those desiring a more extended analysis of the operators A, K, C, N, *etc.* will find the work of H. B. Curry on Schoenfinkel's Calculus worth studying.

Louis O. Kattsoff
Harpur College

Problèmes Stochastiques Posés par le Phenomène de Formation d'une Queue d'Attente à un Guichet et par des Phenomènes Apparentes. By F. Pollaczek. Gauthier-Villars, Paris, 1957. 123 pp. 2,500 fr.

This little book on queueing theory with a single server, written by one of the leading workers in this field, will provide a good introduction to the subject as approached by the author's complex variables method. This approach, while not always as elegant or revealing of probabilistic intuition as those of Kendall,

Lindley, *etc.*, yields explicit results concerning many of the interesting quantities considered in queueing theory, in many cases. The student who wants a well-written example of the use of Laplace transforms, contour integration, *etc.*, in simple probabilistic applications, can find it here.

The author's approach is usually that of solving an integral equation for the Laplace transform of a limiting distribution function, or of solving a recursion relation for the transforms of a sequence of distribution functions, and then of inverting the result. He considers such things as the waiting time of the nth arrival and also the conditional distribution of waiting time, given that there is an arrival at a specified time T. In some cases of the former, an asymptotic (with n) formula and error term are given. He considers also such modifications of the usual setup as that wherein anyone delayed a positive amount by the usual queueing process also suffers an additional delay, and that wherein anyone who encounters a busy queue leaves without service. Joint distributions of several waiting times are considered, as are busy periods. The examples are often in the case of exponential service times and inter-arrival times, but a few other interesting cases are also treated in detail.

J. KIEFER
Cornell University

Elementary Statistical Methods (Rev. Ed.). By Helen M. Walker and Joseph Lev. Holt, New York, 1958. xvi+302 pp. $4.75.

The authors state that this is "a revision of the book under the same name published by the senior author in 1943," but that "it is actually a completely new piece of writing." "The authors have attempted to provide an introduction to statistical inference which is modern, intuitive, and nonmathematical." The book "is written primarily for the person whose study of mathematics ended when he left high school." The general quality of exposition in the text is good and, in general, is at the level intended. Chapter 13, "Introduction to Statistical Inference," impressed the reviewer as particularly clear and precise. The material covered is that widely agreed upon as basic in a first, nonmathematical course, and includes also such topics as Computing the Power of a Test, The Sign Test, Yates' Correction, and The z_r Transformation (of Fisher). The subject of inference is essentially postponed until the descriptive phase of statistics has included regression and correlation. Since the authors strongly advise (viii) that their sequence of topics be followed in "*this* book," those who wish to take up inference in relation to means of normal populations and then proceed to regression and correlation may find difficulty in using this text. The problems appear to be good and in sufficient supply, and answers are given. There is no numbering of sections in chapters. Overall, this should be a satisfactory text for use in a nonmathematical course, following the authors' sequence. There are a few items on the other side. The statement on page 62 concerning the ordinate

of a frequency polygon seems to be contradicted by the statement on page 63. The fourth paragraph on page 168 is completely misleading and irrelevant. On page 197 it is implied that $Z=50+10(X-\overline{X})/s$ is normally distributed if X is so distributed. There are fairly obvious misprints in the text on pages 52, 104, 172, 221; in leftmost column head for X's, page 147; and in (7.1) and (16.7).

FRANKLIN S. MCFEELY
Montana State College

Understanding and Teaching Arithmetic in the Elementary School. By E. T. McSwain and Ralph J. Cooke. Holt, New York, 1958. xi+420 pp. $5.50.

This text deals with instructional methods and, according to the authors (p. v), with the improvement of the teacher's "understanding of the meanings, vocabulary, and mathematical operations that constitute the language and science of arithmetic." The authors' way of pursuing this aim is to present a kind of cookbook of procedural rules supported by numerical examples and by explanations which are consistently unsound or inadequate. It is a case of the blind leading the blind, the result being a confusion too extensive to document here.

There is no indication that the authors understand the distinction between symbols and the referents which they denote. It is hard to tell under just what circumstances the authors regard a fraction as a number, but they are quite definite about zero: Zero is always only a placeholder symbol, never a number, not even a digit, an expression like $0\div 5$ posing an impossible problem. A *number*, we are told in formal definition, is "A mathematical symbol or symbols to show the idea of total amount of a quantity or total units in a group. It possesses place value and face value." There are 121 such definitions listed under the heading "Basic Vocabulary in Arithmetic" on pp. 367–377, of which the following are typical:

Division: "A mathematical and mental process used to change the dividend into numbers like the divisor. . . . " *Multiplicand:* "One of a given number of like numbers to be grouped and the product notated by one number." *Perimeter:* "The sum of the length of the sides of a surface plane. The outer boundary of a plane." *Subtrahend:* "A number in subtraction that identifies one of the given component numbers of a given number." *Time:* "Continuous movement. . . . "

That a text like this one can find a market; that our elementary school texts are often equally mathematically illiterate; for this a large portion of blame must be shouldered by college mathematicians, who by their neglect have left the control of lower school education to the untrained.

ROBERT L. SWAIN
State University Teachers College
New Paltz, New York

Differential Equations Applied in Science and Engineering. By Harold Wayland. Van Nostrand, Princeton, N. J., 1957. xiii+353 pp. $7.50.

This text is written for advanced undergraduates and beginning graduate students in applied science and engineering. The author intends it "to follow a two-year course in calculus and analytic geometry and a basic course in physics . . . ." His primary aim is to provide a rapid introduction to classical tools for solving the partial differential equations of theoretical physics.

The pattern followed is roughly that established some twenty years ago in mathematical books for engineers. Following a brief beginning chapter on problem formulation the author devotes forty-seven pages to a treatment of vector analysis with emphasis upon orthogonal curvilinear coordinate systems and upon coordinate-free vector properties. Chapters III and IV offer some ninety pages on methods for solving linear ordinary differential equations, including solutions by means of power series. Chapter V, on functions defined by differential equations, outlines the properties and applications of Bessel functions, Gamma functions and Legendre functions. A fifty-eight page discussion of Fourier series and orthogonal functions in Chapter VI is followed by forty pages devoted to boundary value problems. The method of separation of variables is applied to standard problems in heat flow, neutron diffusion, wave motion and the like. The final short chapter on integral transforms yields a description of finite Fourier and Hankel transforms, the Fourier transform and Laplace transform.

The author has managed to compress a surprisingly large number of mathematical facts and formulas in his book, including a selection of standard topics from advanced calculus. The informal, conversational style of the author nevertheless makes for pleasant reading without sacrificing pedagogical effectiveness. The reviewer notes with approval the inclusion of a list of some forty-seven selected references to problems in the recent technical literature, illustrating a variety of applications.

From the mathematician's point of view the results are overly informal in statement, proofs are noticeably absent, derivations are brief and heuristic and mathematical niceties and generalities are, in the main, neglected. Indeed, the headlong pace of the book allows slight opportunity to develop a proper appreciation of the power of rigorous mathematical analysis. For example, the discussion of Sturm-Liouville systems together with associated eigenvalue and expansion properties occupies but four pages in the single section on orthogonal functions. For this reason it is all the more regrettable that the chapter bibliographies do not contain many well-known works which offer excellent supplemental treatments.

The applied mathematician, in particular, will be disappointed by the lack of emphasis on initial problem formulation and by the limited attention allotted to interpretation and justification of final results. Also, he will seek in vain for a chapter on methods of numerical and approximate solution which are indis-

pensable for today's applied scientist.

For students in technical courses, preparing to cope with tomorrow's world of science, a knowledge of classical differential techniques is essential. The text under review provides a notably rapid and cursory survey of classical techniques and applications of differential equations. Because of the elementary and introductory character of the text the reviewer heartily endorses the author's recommendation that the student seek supplementary detail and news of modern mathematical developments in the established literature on applied mathematics.

B. H. Colvin
Boeing Scientific Research Laboratories
Seattle, Washington

Finite-Dimensional Vector Spaces (2nd ed.) By P. R. Halmos. Van Nostrand, Princeton, N. J., 1958. 199 pp. $5.00.

The 1943 reviews by E. R. Lorch and M. Kac of the first edition (denoted by FE) are made part of this review, for most of FE has been copied directly (occasionally with original misprint: "Negative" p. 120). Added are: (1) Fields, (2) Determinants, (3) Problems, (4) Other.

Fields: Usually not of characteristic 2, occasionally algebraically closed; these blanket assumptions make many theorems not self-contained in their statements; later in the book the field has become real or complex. As usual, one with no knowledge of fields may use real or complex scalars throughout, without losing any of the spirit of the book. (Paradoxically, finite dimensionality is never a blanket assumption.)

Determinants: These are scalars associated with operations (necessarily multiplication by a scalar) on the space (necessarily 1 dimensional) of alternating multilinear forms of highest degree. Clearly these coordinate-free methods (already the theme of FE) make this a mathematician's book as opposed to the recent plethora of "educational" texts on matrices.

Problems: Excellent! Ranging from elementary checks to a casual request for proof of the von Neumann-Jordan characterization of inner-product space. However the author announces a dangerous policy: declarative problems may be false! Fortunately, he seems to have reconsidered. Almost without exception the problems mean what they say, and in one case the phrase "true or false" is used.

Other changes: The elegant proof of Schwarz's inequality via Bessel has been borrowed from the author's book (denoted by HS) on Hilbert space; fine if one can postpone the triangular inequality so long. Tensor product is dual of space of bilinear forms on direct product; complexification is done this way and, later, directly.

Paradoxically, the proof of invertibility (if $|1-A|<1$) is more general (after omission of a redundant step, copied from FE!) than the one in HS.

With only the generalization to Hilbert space in mind, the book uses at

several points far more than the minimum needed. To give two examples: In the proof that a linear transformation must be bounded; in the Riesz representation theorem, much of which can be phrased so as to hold in a uniformly convex normed space.

Of 9 misprints noted, the most annoying is $\overline{BA}$ for $\overline{B}\overline{A}$, page 99. (Three times!) Foreign mathematicians are warned not to search in dictionaries for zeroish, zeroness, askable, ... In style, Professor Halmos follows G. H. Hardy in the role of a "missionary preaching to the cannibals."

ALBERT WILANSKY
Lehigh University

Engineering Mathematics. By Robert E. Gaskell. Holt, New York, 1958. xvi+462 pp. $7.25.

Here is another text on the ever-popular subject of advanced mathematics for engineers, this one by the head of a mathematical research laboratory of Boeing Aircraft. The traditional material is covered: differential equations, matrices, complex variables, Fourier series, numerical methods, special functions, vector analysis, Laplace transforms, boundary value problems. The methods used and physical applications made are, for the most part, the thrice familiar ones but there are praiseworthy efforts at timeliness. Rigor is average for its category and, when lacking, there is no pretence. A first-rate reference is usually furnished for omitted proofs or developments. It goes without saying that this manner of text must perforce emphasize time-tested ingredients. The individual teacher can venture farther at his own risk.

There are brief sections which are mildly and estimably novel: Dimensional analysis, analog computation, complex integration, least squares, automatic control. Uncluttered typography, simple figures, and a distaste for nonessential complications makes this book unformidable in appearance and relatively easy to absorb. Clarity is a dominant characteristic. There are plenty of exercises but the fact that the answers are given may not appeal to all teachers. Most unusual is the intelligence of the author's comments in introducing each new subject. Even the preface is a pleasure to read.

One must always have reservations. It can be argued that too much effort is devoted to the D-symbolic method for linear differential equations. More appropriate might have been the succinct all-purpose weighting function method so popular with engineers. The unit impulse, now a universal tool, is conspicuously absent, as are Fourier transforms. Graeffe's method, now that digital computers are rampant, should yield to Lin.

Complete lucidity and the accepted norm of comprehensiveness seem to be the goals set by the author and they are achieved. Just about all the standard basic material is here and in highly digestible form. A knowledgeable instructor will find it flexible enough to permit exercise of his own personality as well.

JOHN L. VANDERSLICE
Applied Physics Laboratory, The Johns Hopkins University

Essential Business Mathematics. Llewellyn R. Snyder. McGraw-Hill, New York, 1958. 470 pp. $5.50.

This book, which has been designed as a textbook, is divided into two parts. The first 105 pages are devoted to grade school arithmetic for the reader who is deficient in the fundamental skills of computation and problem solving. The second part of the book is devoted to such topics as simple and compound interest, periodic payment plans, insurance, taxes, corporate securities, profits, discount and markup, problems in retailing, stocks and bonds, and annuities. The author sticks religiously to his original intent to present the topics of his book with no mathematics beyond elementary arithmetic. In the development of annuity formulae he avoids geometric progressions by supplying the end result of progressions as applied to the compound interest problem and by accentuating the use of tables. The author presents much interesting and helpful information on effective interest rates, incentive wage formulae, social security, corporate securities, and stocks and bonds.

It is the reviewer's opinion that this book would more appropriately lend itself to review in a business periodical. It is more *business* than it is *mathematics*. There is a preponderance of devotion to "rules of thumb" rather than to the critical thinking which makes the student somewhat independent in the solution of financial problems. For a text claiming to be very practical there appears to be too much emphasis on short cuts in computation, "casting out nines," and aliquot parts tables. The text does *not* contain enough mathematics to form a sound basis for accounting, business finance, and statistics as claimed by the author. It might make a text for a general business course or a reference source for a small businessman.

TRUMAN WESTER
Central State College

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

MIDWESTERN CONFERENCE ON MECHANICS

The Sixth Conference on Fluid Mechanics and the Fourth Conference on Solid Mechanics will be held at the University of Texas on September 9–11, 1959. A series of invited lectures related to the program is being sponsored by the University of Texas. These will include the following: "Recent Progress in Rarefied Gas Dynamics Research" by Professor S. A. Schaaf, University of California, Berkeley; "New Developments in Shell Theory" by Professor L. H. Donnell, Illinois Institute of Technology; "Recent Progress in Applied Mechanics" by Professor Sydney Goldstein, Harvard University.

Correspondence regarding details of the Conference and technical papers should be sent to: Dr. M. J. Thompson, Chairman, Midwestern Conference on Fluid and Solid Mechanics, Department of Aeronautical Engineering, The University of Texas, Austin 12, Texas.

PERSONAL ITEMS

Mr. L. A. Shepp, Brooklyn College, has won the William Lowell Putnam Prize Scholarship for the eighteenth competition.

National Bureau of Standards: The following professors of mathematics have been awarded fellowships in Numerical Analysis for four-month periods; R. V. Andree, University of Oklahoma; T. A. Botts, University of Virginia,; G. C. Byers, Michigan College of Mining and Technology; R. T. Gregory, University of Texas; R. E. Lee, University of Missouri; L. F. Meyers, Ohio State University; E. P. Miles, Jr., Florida State University; B. C. Moore, Agricultural and Mechanical College of Texas; J. D. Munn, Mississippi Southern College.

Mr. R. S. Barton, Shell Development Company, Houston, Texas, has been appointed Manager of Applied Programming, The Electrodata Division of Burroughs Corporation, Pasadena, California.

Associate Professor J. W. Carr, III, University of Michigan, has been appointed Associate Professor and Director of the Research Computation Center, University of North Carolina.

Dr. C. E. Clark, Booz, Allen and Hamilton, Chicago, Illinois, has accepted the position of Mathematician with the Systems Development Corporation, Santa Monica, California.

Assistant Professor R. L. Davis, University of Virginia, has been awarded a Science Faculty Fellowship by the National Science Foundation for the academic year 1959–60 and will study at Stanford University.

Dr. Alice B. Dickinson has been appointed a Lecturer at Smith College.

Professor J. A. Dieudonne, Northwestern University, has accepted appointment as a permanent member of the staff of a new French research institute in Paris.

Assistant Professor H. A. Heckart, Simpson College, has been promoted to Associate Professor.

Professor F. F. Helton, Central College, Fayette, Missouri, has been appointed Professor at the College of the Pacific.

Dr. T. R. Horton, International Business Machines Corporation, has been appointed Manager of 700–7000 Systems Marketing, Data Processing Division, White Plains, New York.

Mr. H. H. Hunt, Oklahoma State University, has been appointed Instructor at Central State College, Edmond, Oklahoma.

Professor J. L. Katz, Rensselaer Polytechnic Institute, has been awarded a Science Faculty fellowship for a year's study at University College, London, England.

Professor D. M. Krabill, Bowling Green State University, has been awarded a National Science Foundation Science Faculty Fellowship for study in the Operations Research Program at the University of Michigan during the summers of 1959–61.

Mrs. Marilyn J. Lockhart, Programmer at International Business Machines Corporation, Poughkeepsie Research Laboratory, has been appointed Associate Mathematician in the Computing Department.

Dr. M. M. Lotkin has accepted the position of Principal Staff Scientist with Avco Research Division, Wilmington, Massachusetts.

Mrs. Ann S. Miller, Roanoke Catholic High School, Virginia, has been appointed Junior Engineer at the Lycoming Division, Avco Manufacturing Corporation, Stratford, Connecticut.

Mr. Otto Mond, Army Signal Engineering Laboratory, has accepted the position of Technical Engineer with the Aircraft Nuclear Propulsion Department, General Electric Company, Cincinnati, Ohio.

Mr. Joel Niedelman, Burroughs Corporation, Paoli, Pennsylvania, has accepted a position as Mathematician with the Bendix Aviation Corporation, North Hollywood, California.

Mr. C. E. Prince, Jr., Taejon Presbyterian College, Taejon, Korea, has been promoted to Assistant Professor.

Mr. M. B. Richins, Arizona State College, has accepted the position of Engineer with the Bell Telephone Laboratory, Murray Hill, New Jersey.

Mr. I. S. Rubin, Alwac Corporation, New York, has accepted the position of Sales Engineer with the Thomson-Ramo-Wooldridge Products Corporation, Los Angeles, California.

Mr. F. C. Sherburne, Jr., Michigan State University, has been appointed Instructor at Hope College.

Mr. W. W. Shirley, Richfield Oil Corporation, Los Angeles, California, has been appointed Mathematician-I.B.M. Programmer.

Captain M. J. Stager, Agricultural and Mechanical College of Texas, has been appointed Instructor at the United States Air Force Academy, Colorado.

Mr. Melvin Tainiter, Fairchild Astrionics Division, Wyandauch, Long Island, has accepted the position of Associate Engineer with the Arma Division, American Bosch Arma Corporation, Garden City, New York.

Professor Emeritus W. E. Cleland, Geneva College, died on February 26, 1959. He was a member of the Association for 39 years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

Professor H. M. Gehman, Secretary Treasurer, announces that the following 159 persons have been elected to membership by the Board of Governors on applications duly certified.

IRVING ADLER, M.A. (Columbia) Instr., Columbia University.

CARL M. ALBERT, A.B. (Brown) Grad. Student, Quinnipiac College.

ROGER W. ALDRICH, JR., Student, Drew University.

DONALD W. ANDERSON, Student, California Institute of Technology.

GLORIA D. ANDREWS, B.S. (Roosevelt) Teacher, Carver High School, Chicago, Illinois.

WILLIAM P. BAIR, M.A. in Ed. (Occidental) Teacher, Pasadena City Schools, California.

EUGENE H. BALSTER, B.S. (S. Dakota S.C.) Research Engineer, North American Aviation.

ERWIN J. BASINSKI, B.A. (St. Thomas) Grad. Student, University of Houston; IBM Programmer, Shell Chemical Corp.

DAVID S. BATES, Student, Butler University.

DONALD BATMAN, Student, San Francisco State College.

H. LYNNE BERNSTEIN, B.S. (Illinois Inst. of Tech.) IBM Operator, Federal Bank of Chicago.

JOHN A. BERTON, M.A. (Illinois) Asst., University of Illinois.

NORMAN BLEISTEIN, Student, Brooklyn College.

MRS. PATRICIA L. BRANDT, B.S. (Oklahoma) Head of Dept., U. S. Grant High School, Oklahoma City, Oklahoma.

HAZEL H. BREWER, M.S. (Southern California) Teacher, Pasadena Board of Education, California.

CHARLES K. BROWN, III, M.S. in Ed. (Cornell) Head of Dept., Westtown School, Pennsylvania.

WALTER D. BURGESS, Student, University of British Columbia.

CLARENCE E. BUTLER, M.Ed. (Maine) Chairman of Dept., Wilton High School, Connecticut.

WILLIAM G. CALDWELL, B.S. (U.S. Military Acad.) Teacher, San Francisco City College.

JOHN W. CALVERT, Student, University of Kentucky.

CONRAD E. CAMPBELL, M.A. (Syracuse) Grad. Asst., Syracuse University.

MICHAEL E. CHAMBREAU, Student, Stanford University.

GEORGE F. CHAPLINE, JR., Student, University of California.

DOROTHY J. CHRISTENSEN, Ph.D. (Washington) Instr., Wellesley College.

RALFE J. CLENCH, JR., B.A. (Queen's, Canada) Part time Lecturer and Grad. Student, Queen's University, Canada.

LEON COHEN, Student, City College of New York.

CARLTON J. COOK, B.S. (M.I.T.) Grad. Student, Harpur College.

WILLIAM F. COULSON, B.S. (Iowa S.C.) Teacher, Bellflower High School, California.

CAROL A. CUNNINGHAM, Student, Butler University.
MARY M. DAMARODIS, Student, Seton Hall University.
PHILIP J. DAVIS, Ph.D. (Harvard) Chief, Numerical Analysis, National Bureau of Standards.
JOHN G. DEMAS, Midshipman, United States Naval Academy.
GEORGE G. DEN BROEDER, JR., M.A. (Wayne S.U.) Instr., University of California; Research Specialist, Lockheed Missiles & Space Division.
PETER A. DIAMOND, Student, Yale University.
MRS. ROSIE L. EDWARDS, B.S. (Wiley) Teacher, Bruce High School, Gilmer, Texas.
ROBERT J. ELLIOTT, Student, New College, Oxford, England.
ERNEST G. ENNS, Student, University of British Columbia.
LAVERNE R. ESPELAND, B.S. (Augustana) Grad. Student, University of Colorado.
ROLFE P. FERGUSON, Student, Hamilton College.
BROTHER AUSTIN G. FITZGERALD, F.S.C., M.A. (Manhattan) Teacher, De La Salle College; Grad. Student, Catholic University.
DANIEL M. FOLEY, JR., B.S. (Tulane) Grad. Asst., Tulane University.
MRS. MARJORIE L. FRENCH, M.S. (Kansas) Chairman of Dept., Topeka Public Schools, Kansas.
BERT E. FRISTEDT, Student, University of Minnesota.
DONALD W. GARLOCK, Student, George Washington University; Research Asst., Resources for the Future.
JAMES R. GEISER, Student, Massachusetts Institute of Technology.
DANIEL P. GIESY, Student, Ohio State University.
EDWARD S. GINSBERG, Student, Brown University.
GRETCHEN B. GLASS, Student, University of Oregon.
GEORGE GLAUBERMAN, Student, Polytechnic Institute of Brooklyn.
OLGA L. GLENN, B.S. (Roosevelt) Teacher, Carver High School, Chicago, Illinois.
JAY R. GOLDMAN, Student, Brooklyn College.
ALFRED GRAY, JR., Student, University of Kansas.
MRS. MARY S. GREEN, M.A. (Western Reserve) Instr., Taylor University.
PHILLIP A. GRIFFITHS, Student, Wake Forest College.
CHARLES D. GRIMSRUD, B.A. (Concordia) Grad. Asst., University of Nebraska.
FLETCHER I. GROSS, Student, California Institute of Technology.
ALFRED W. HALES, Student, California Institute of Technology.
HENRY HEDDEN, M.Ed. (Georgia) Instr., Young Harris College.
CECIL A. HEICK, Student, University of Idaho.
PAUL L. HEITNER, Student, Brooklyn College.
MRS. SONIA HENCKEL, Student, University of Wisconsin.
KENNETH D. HERR, B.S. in Ed. (Pennsylvania S.T.C., Millersville) Grad. Student, University of Illinois.
CHARLES H. HONZIK, Ph.D. (California, Berkeley) Psychologist, Veterans Administration, San Francisco, California.
WILLIAM A. HORN, Student, University of Cincinnati.
J. LAMAR JENSEN, M.A. (Utah) Instr., Weber College.
M. ROBERTA KEITER, Ph.D. (Maryland) Teacher, Montgomery Blair High School, Silver Spring, Maryland.
JOHN W. KENELLY, JR., M.S. (Mississippi) Grad. Asst., University of Florida.
WILLIAM L. KENT, Student, Pomona College.
WILLIAM E. KERBY, Student, University of Colorado.
JO ANN M. KERR, Student, University of Oklahoma.
WILLIAM E. KESLER, B.S. (Florida Southern) Computer, Boeing Airplane Co.
LEON KOTIN, Ph.D. (New York) Mathematician, U.S. Army Signal Research & Development Laboratory.
KENNETH B. KRIEGE, M.A. (California S. Poly.) Teacher, California State Polytechnic College.
G. LAWRENCE LANE, B.A. (St. Benedict's) Grad. Asst., University of Kansas.
MRS. ANNELI LAX, Ph.D. (New York) Part time Instr., New York University; Technical Editor, S.M.S.G., Yale University.
BROTHER EDWARD LEE, C.S.C., B.S. (St. Edward's) Head of Dept., Holy Cross High School, New Orleans, Louisiana.
FREDERICK J. LEIPOLD, Student, Purdue University.
FREDERICK W. LEYSIEFFER, M.A. (Wisconsin) Teaching Fellow, University of Michigan.
CARMEL A. LING, M.S. in Ed. (Southern California) Teacher, Excelsior Union High School District, Artesia, California; Psychological Consultant, Whittier, Calif.
MRS. JEANETTE E. LOCKLEY, M.S. (Texas Southern) Instr., Oakland Junior College.
TOM E. LOK, Student, Oregon State College.
STANLEY I. MACK, B.S. (Brooklyn) Teaching Asst., Syracuse University.
MICHAEL C. MACKEY, Student, Wyandotte High School, Kansas City, Kansas.
JACK W. MACKI, Student, University of Idaho.
RONALD J. MACKINNON, Student, St. Francis Xavier University, Canada.
WAYNE L. MCDANIEL, B.S. (Northwest Missouri S.C.) Asst. Instr., University of Missouri.
THOMAS M. MIKULA, M.A. (Columbia) Instr., Phillips Academy.
HENRYK MINC, M.A. (Edinburgh) Lecturer, University of British Columbia.
JAMES P. MINNEBO, B.S. (Detroit) Teaching Fellow, University of Detroit.
PARVIZ MOREWEDGE, B.A. (U.C.L.A.) Computing Analyst, Douglas Aircraft Co.
NORMAN G. MOUCK, JR., M.Ed. (U.C.L.A.) Chairman, Science, Dept., Fontana High School, California.
DAVID L. MURRAY, B.A. (New York) Field Engineer, Western Electric Co.
MARILYN A. MYERS, Student, San Jose State College.
ALBERT M. NAGY, M.A. (Montclair S.C.) Head of Dept., Pomfret School, Connecticut.
DAVID S. NEWMAN, B.S. (New Mexico) Teaching Asst., Cornell University.
ALBERT NIJENHUIS, Ph.D. (Amsterdam) Asso. Professor, University of Washington.
JAMES M. O'CONNELL, Student, University of Wisconsin.
CHARLES L. ODOROFF, Student, Carleton College.
WILLIAM M. O'FALLON, M.A. (Vanderbilt) Instr., St. John's University.
STEWART OGDEN, A.B. (Harvard) Engineer, Raytheon Manufacturing Co.
MARK M. OKIMOTO, B.A. (Hawaii) Technical Staff, Hughes Aircraft Co.
CHARLES F. OSGOOD, Student, Haverford College.
CLARENCE A. OSTER, M.S. (Oregon) Asso. Research Engineer, Jet Propulsion Laboratory.
ROBERT R. PARKER, B.S.E.E. (Mississippi S.C.) Engineer, General Electric Co.
WYN R. PAULY, B.A. (Pacific Union) Mathematician, United States Air Force.
NORMAN L. PAVLIK, M.S. (Indiana) Instr., Union-Whittier Community School, Iowa.
JOHN W. PENNISTEN, II, Student, Hamilton College.
MORRIS POLLCHIK, M.A. (Denver) Teacher, Downey Senior High School, California.
ROBERT R. POOLE, Student, Pomona College.
RICHARD D. PORTES, Student, New Trier High School, Winnetka, Illinois.
IKIE N. PRESSON, B.S. (Western Carolina) Grad. Asst., University of Kentucky.
JOHN N. RAINWATER, A.B. (Washington) Grad. Student, University of Washington.
CLIFFORD O. RANHEIM, M.A. (Minnesota) Asst. Professor, Concordia College.
MAURICE J. REINHART, M.Ed. (Illinois) Teacher, Castlemont High School, Oakland, California.
FAZLOLLAH REZA, Ph.D.E.E. (Brooklyn Poly. Inst.) Professor, Syracuse University.
JACK W. RHOADS, Student, San Diego State College.
EDWARD S. ROBBINS, M.A. (Wichita) Asst. Professor, Fresno State College.
JOHN A. ROBBINS, Student, Swarthmore College.
LESTER A. RUBENFELD, Student, Brooklyn Polytechnic Institute.
ROBERT E. RUNDUS, M.Ed. (Illinois) Teacher Asso., UICSM, University of Illinois.
LEROY SACHS, M.S. (Washington, Missouri) Chairman of Dept., Cahokia High School, St. Louis, Missouri.
PAUL L. SADAGURSKY, B.S. (Queens) Instr., Academy of Aeronautics, New York.
PAUL F. SAMPSON, Student, Iowa State College.
JEFFREY D. SCARGLE, Student, New Trier High School, Winnetka, Illinois.

BRUCE E. SCHROEDER, B.A. (Iowa S.T.C.) Teacher, Garwin High School, Iowa.
MRS. DOROTHY R. SEARCY, B.S. (Virginia S.C.) Teacher, Carver High School, Chicago, Illinois.
LAWRENCE F. SHAMPINE, Student, California Institute of Technology.
E. MAE SHENK, A.B. (Hood) Chairman of Dept., Hempfield High School, Pennsylvania.
LAWRENCE C. SHEPLEY, Student, Swarthmore College.
RALPH T. SHUEY, Student, New Trier High School, Winnetka, Illinois.
SISTER MARY BONAVENTURE, C.S.S.F., A.B. (Villanova) Teacher, Immaculate Conception College.
SISTER MARY DE PORRES LYNCH, M.S. (Notre Dame) Chairman of Dept., St. Agnes High School, Rochester, New York.
SISTER MARY FLORENCE BROWN, R.S.M., M.A. (Catholic) Dean, Mercy Junior College.
SISTER MARY OSWALD WALSH, M.A. (Boston) Instr., College of Our Lady of the Elms.
LARRY L. SLEIZER, Student, University of Washington.
DON R. SMITH, Student, University of Oklahoma.
ELIZABETH D. SMITH, M.A. (Columbia) Head of Dept., Milford High School, Connecticut.
DAVID A. SPRECHER, B.A. (Bridgeport) Grad. Asst., University of Maryland.
MARK E. STERN, M.A. (Johns Hopkins) Applied Science Representative, I.B.M. Corp.
MRS. GRACE E. STRICKLER, A.B. (San Jose S.C.) Teacher, James Lick High School, San Jose, California.
ETHEL L. STUBBLEFIELD, M.A. (California) Math. Consultant, Fresno City Unified School District, California.
YOZO TAKEDA, B.S. (Michigan) Asst. Instr., University of Missouri.
HOWARD I. TEICH, Student, Polytechnic Institute of Brooklyn.
ROBERT F. TEITEL, Student, Massachusetts Institute of Technology.
DONALD H. THOMAS, B.S. (Wayne S.U.) Grad. Asst., Wayne State University.
PRENTISS F. TOMLINSON, M.A. (Ohio S.U.) Instr., Olivet Nazarene College.
HAROLD L. TOOTHMAN, B.S. (Geo. Washington) Mathematician, Naval Research Laboratory.
BRUCE E. TRUMBO, Student, Knox College.
JOHN J. UCCI, Student, Stevens Institute of Technology.
RICHARD L. VAN DE WETERING, M.Ed. (Western Washington) Acting Instr., Stanford University.
VINCENT G. VITALE, M.A. (Kent S.U.) Instr., Fordham University.
RICHARD C. WADLE, B.A. (Baylor) Grad. Asst., Baylor University.
WALDEMAR C. WEBER, Midshipman, United States Naval Academy.
ARTHUR T. WHITE, II, Student, Oberlin College.
REV. ANDREW P. WHITMAN, S.J., M.S. (Catholic) Grad. Student, Catholic University.
HOWARD L. WIENER, Student, University of Oregon.
JANICE E. WILSHIRE, Student, Baylor University.
LOWELL S. WINTON, Ph.D. (Duke) Professor, North Carolina State College.
MOON WON, B.A. (Dangook) Head of Dept., Korean Military Academy, Seoul.

REQUESTS FOR PUBLICATIONS OF CUP

The Association's Committee on the Undergraduate Program in Mathematics has arranged for the printing and distribution of the following five books:

UNIVERSAL MATHEMATICS
ELEMENTARY MATHEMATICS OF SETS—WITH APPLICATIONS
MODERN MATHEMATICAL METHODS AND MODELS, VOLUME I: MULTICOMPONENT METHODS
MODERN MATHEMATICAL METHODS AND MODELS, VOLUME II: MATHEMATICAL MODELS
CALCULUS AND ANALYTIC GEOMETRY, BY EMIL ARTIN

The contents of these books are described in advertisements printed in this MONTHLY, August–September 1958 and October 1958.

Since the demand for these books has exceeded expectations, it has been necessary to have them reprinted, and accordingly their distribution has been greatly delayed. It is expected, however, that by the time this notice appears, current orders will be filled promptly. If, therefore, your request for any of the above books was mailed before May 1 and has not yet been filled, please send a second request to the Mathematical Association of America, University of Buffalo, Buffalo 14, New York. A nominal charge will be made for future requests.

HARRY M. GEHMAN, *Secretary Treasurer*

PROGRAM OF VISITING LECTURERS, 1959–60

With the financial support of the National Science Foundation, the Visiting Lectureship Program administered by the MAA since 1954 will be continued during the academic year 1959–60. Lecturers and the regions covered are as follows:

Professors J. S. Frame of Michigan State University and S. A. Jennings of the University of British Columbia will lecture in the Western Mountain and Pacific region.

Professors David Blackwell of the University of California, Berkeley, M. R. Hestenes of the University of California, Los Angeles, and Ernst Snapper of Indiana University will lecture in the Central and Gulf region.

Dr. Philip J. Davis of the National Bureau of Standards and Professor Tibor Rado of Ohio State University will lecture in the Eastern region.

A booklet describing the Program of Visiting Lecturers may be obtained by writing to the office of the Association at the University of Buffalo.

THE MARCH MEETING OF THE SOUTHEASTERN SECTION

The thirty-eighth annual meeting of the Southeastern Section of the Mathematical Association of America was held March 20–21, 1959, at East Tennessee State College, Johnson City, Tennessee. Professors D. E. South, Chairman of the Section; T. C. Carson, Vice-Chairman; M. C. Boyce; B. G. Clark; F. A. Ficken; F. A. Lewis; and C. L. Seebeck, Jr., presided over the general and divisional sessions. There were 210 in attendance, including 140 members of the Association..

The following officers were elected for the coming year: Chairman, Professor T. C. Carson, East Tennessee State College; Vice-Chairman, Professor T. H. Lee, University of South Carolina; Secretary-Treasurer, Professor C. L. Seebeck, Jr., University of Alabama. At the business meeting Professor E. D. Nichols, Florida State University, gave a report of the High School Mathematics Contest as held in Florida. Mostly negative reports were received from the other states of the section, since many conduct their own contests.

After the Friday evening banquet, a silver tray was presented to the retiring secretary in appreciation of his service as Secretary-Treasurer of the Section since 1933.

The following program was presented:

1. *A note on spherical graphs*, by Professor W. G. Miller, Clemson College.

The paper was presented essentially as an instructional device and noted one aspect of an investigation relating to curves of infinite extent. Using a hybrid system of rectangular and polar coordinates, the infinite Euclidean plane was mapped on a sphere. Cartesian abscissas and ordinates were taken analogous to geographical longitudes and latitudes. The transformation of units was attained with successive partial sums of a geometric series, converging from zero at one pole to infinity at the other. Results were demonstrated with several graphs on a sphere of circumference 1000 mm., taking $S=500$, $a=25$, and $r=0.95$.

2. *Deciphering infinite messages*, by Dr. Basil Gordon, Army Ordnance Missile Command, Redstone Arsenal, Alabama.

Let $A=\{1, \cdots, n\}$, and let M be an infinite sequence $\cdots a_{-2}a_{-1}a_0a_1a_2 \cdots$, with a_i in A. Finally, let D be a fixed collection of "words" $(c_1, \cdots, c_k)$, c_i in A. The message M is said to be decodable if it can be divided into words of D by insertion of parentheses. If all decodable messages are uniquely decodable, the size of D is severely restricted. Inequalities on the size of D are established. The relation of these inequalities to the results of McMillan and Shannon on finite messages is discussed.

3. *Rocket trajectories—series solutions*, by Professor R. C. Meacham, University of Florida.

To investigate the degree of polynomial needed for real-time smoothing procedures during rocket flight, power series solutions of the equations of motion may be used. Such solutions are presented under various conditions of rocket flight, including spherical earth gravity field, constant altitude thrust force, "gravity turn" thrust force, and drag force.

4. *Method of orbit determination for an artificial earth satellite*, by Dr. J. R. Garrett, Rich Electronic Computer Center, Georgia Institute of Technology.

The motion of an artificial earth satellite is given by a system of three nonlinear differential equations. The equations are formulated to account for the oblateness of the earth and the atmospheric resistence. Methods of numerical integration for the system of equations are discussed. Some of the difficulties involved in precision orbit work is mentioned. The research effort currently in progress at the Rich Electronic Computer Center is the basis for comment.

5. *Retired officer program*, by Professor J. J. Gergen, Duke University.

This program is supported at Duke University by the National Science Foundation and is designed to retrain and prepare participating retired armed services officers to teach high school mathematics or college mathematics through calculus. The program leads in one year to the degree of Master of Arts in Teaching.

6. *On the number of trials necessary to produce a success*, by Professor D. E. South, University of Florida.

This paper contains a short résumé of the history of the Theory of Probability, particularly with reference to problems with varying trial probabilities. Two cases for the probability that exactly $x+r$ trials will be required to produce r successes are discussed. In the first case, the trial probability depends upon the number of previous successes, and in the second, the trial probability is a function of the number of the trial.

7. *NSF summer mathematics camp for talented high school students*, by the Director, Professor E. D. Nichols, Florida State University.

Forty mathematically-talented high school students from ten states were selected from 1100 applicants to attend The Florida State University 1958 six weeks Summer Mathematics Camp sponsored by the National Science Foundation. For about four hours daily, the students studied mathematics not usually found in the high school program. In addition they had a Russian language course, and listened to a number of lectures given by scientists from various fields. A similar camp will be held June 15 through July 24, 1959.

8. *Teaching of trigonometry on closed circuit TV*, by Professor Ayrlene M. Jones, University of Alabama.

The experiment of the University of Alabama in teaching trigonometry by closed-circuit television is described. It was found that no appreciable difference in grades resulted. Students preferred this type of instruction with an experienced teacher to small classes with an inexperienced teacher. Among other benefits noted, graduate assistants who conducted discussion TV classes learned much about teaching by observing and associating with a superior teacher.

9. *Teaching of mathematics of finance on closed circuit TV*, by Professor J. O. Reynolds, East Carolina College.

At East Carolina College this is a service course for the Business Education Department. An elementary algebra test was used to select about 40% of the better students. Daily lectures of a half period were given on TV; then a work session under a supervising instructor completed the period. Overall results were encouraging. The better students did quite well, and results were about the same as obtained in a regular classroom situation for all students.

10. *Help for the neglected student in mathematics*, by Professor W. L. Williams, University of South Carolina.

In this paper the author defined the neglected student in college mathematics today as the gifted student. He pointed out what is being done for the poorly prepared student, such as providing remedial classes for him, but emphasized that little is being done for the student of superior

ability. He suggested among other things that high schools should be encouraged to make available so-called college algebra and trigonometry to their better students so that when they enter college they can begin immediately the study of analytic geometry and calculus.

11. *The superior student program in mathematics and physics and the honors program in engineering*, by Dr. L. S. Winton, North Carolina State College.

This paper is concerned with the relation of the freshman-sophomore superior student program to the junior-senior honors program in engineering at North Carolina State College and with the content of course material. This four-semester program for the competitively-selected personnel of this group consists of a brief review of algebra and trigonometry followed by 42 weeks of integrated analytic geometry and vector calculus and ten weeks in the elementary theory of ordinary differential equations.

12. *The probability distribution of the product of n random variables*, by Professors Richard Schulz-Arenstorff and J. C. Morelock, King College, read by Professor Morelock.

This paper presents the probability distribution of the product of n random independent variables distributed with uniform density over a unit interval. As expected, the distribution becomes more platykurtic with increasing n. It is of particular interest to observe that standard methods of complex integration using characteristic functions centers all attention to one singular point at the origin where the residue is easily summed to obtain the solution.

13. *The probability that* $(n, f(n))$ *is r-free*, by Professor P. J. McCarthy, Florida State University.

An integer is said to be r-free if it is not divisible by the rth power of any prime. Let f be a function from the positive integers into the nonnegative integers. Let $f^*(x)$ be the number of y such that $f(y)=x$. Assume that f^* is finite and that f and f^* are nondecreasing, $f(x)=O(x)$ and $f^*(f(x))=O(x)$. It is then shown that if n is chosen at random the probability that $(n, f(n))$ is r-free is $1/\zeta(2r)$.

14. *On error bounds for Simpson's rule and the trapezoidal numerical integration formula*, by Professor C. L. Seebeck, Jr., University of Alabama.

To offset the deficiency in current calculus texts, several elementary derivations of error bounds for the trapezoidal law and Simpson's rule are given.

15. *The numerical solution of certain parabolic equations using finite differences*, by Dr. F. J. Witt, Oak Ridge National Laboratory.

The parabolic equation $\eta_t(x, t)=f(x, t)$ where $f(x, t)=A(x, t)\eta_{xx}(x, t)+B(x, t)\eta_x(x, t)+C(x, t)\eta(x, t)+D(x, t)$ with initial conditions and two-point boundary conditions may be solved by setting $(\eta(x, t+\Delta t)-\eta(x, t))/\Delta t=Pf(x, t+\Delta t)+Qf(x, t)+Rf(x, t-\Delta t)$, ($P, Q, R$ constants) and solving the resulting differential equations by a central difference method. The solution has been accomplished with fourth order accuracy in Δx on an electronic digital computer and a study made of the choice of P, Q, and R which produces the more accurate and converging solutions. A discussion of truncation errors, validity conditions and mesh spacing is also given.

16. *On mathematical models*, by Dr. O. E. Taulbee, Lockheed Aircraft Corporation, Marietta, Georgia.

In operations research studies the mathematical model is an attempt to simulate the process under consideration, whether existing or proposed. The pertinent features as well as the advantages and disadvantages of mathematical models are presented. Their relationship to a mathematical system is also indicated.

17. *Polynomial solutions of a class of Riccati equations*, by Professor R. W. Cowan, University of Florida.

The Riccati equation is taken in a form that has polynomial coefficients. If the equation is to have one or more polynomial solutions, at least two of the highest degree terms must be equal after the polynomial solution has been substituted in the equations. On this basis it is possible to develop criteria which limit the possible degrees of polynomial solutions. Several examples are given for the various cases, one containing polynomial solutions of three different degrees.

18. *The condition of matrices*, by Dr. C. T. Fike, Oak Ridge National Laboratory, read by title.

Turing's "condition number" $\|A\| \cdot \|A^{-1}\|$ measures the effect on A^{-1} of changes in A. A similar index of stability sometimes can be defined for characteristic roots of matrices. A relation between these two problems can be exhibited.

19. *Effect of a crack on a stress concentration in an orthotropic plate*, by Professor C. B. Smith, University of Florida.

A plate having two perpendicular axes of elastic symmetry in its plane is said to be orthotropic. A plate of this type with a small crack in its interior is considered. The plate is assumed to be in a state of plane stress due to stresses produced by a point load acting near the crack. The solution is obtained by the methods of complex variables, and is quite different from the solution for the corresponding problem for the isotropic plate.

20. *Remarks on the simplex method of linear programming*, by Professor F. A. Ficken, University of Tennessee.

This paper presents a theoretical discussion of the simplex method that is believed to have certain advantages. The problem is prepared beforehand in such a way that the first tableau has certain specific characteristics. It is then proved inductively, even in the degenerate case, that succeeding tableaux have these same characteristics, with strictly increasing values of the functional being maximized; the effectiveness of the method is then clear. The solution of the dual problem, finally, may be read off at once from the marginal entries at the bottom of the leftmost columns of the final tableau, without any renumbering of indices.

21. *High precision calculation of Arcsin x, Arccos x and Arctan x*, by Dr. I. E. Perlin, Rich Electronic Computer Center, Georgia Institute of Technology.

A polynomial approximation for Arctan x in the interval $|x| < \tan(\pi/24)$ is developed. The approximating polynomial consists of nine terms and yields an approximate value accurate to twenty decimal places. The domain $\tan(\pi/24) \leq |x| < \infty$ is subdivided into six intervals. For each of the first five intervals, the addition formula reduces the problem of calculating Arctan x to that of computing Arctan t_i by means of the polynomial approximation. For the final interval, the calculation of Arctan x is made by Arctan $|x| = \pi/2 - \text{Arctan}\,(1/|x|)$. The Arcsin x and Arccos x are computed by means of relationship to Arctan x.

22. *The effects of singular collineations*, by Professor B. G. Clark, Vanderbilt University.

It was observed that in elementary studies of collineations the singular ones are usually dismissed with the statement that there is a certain singular subspace of points which have no correspondent, while the remaining points are carried into another fixed subspace. This note pointed out that there is induced in the fixed subspace a collineation which may be of any type, and proved the general theorem: *If in a space S_n the matrix of a collineation has rank $n-k$, there is a singular space S_k and all other points of S_n are carried into a fixed S_{n-k-1}; all points of the S_{k+1} determined by the singular S_k and a point B, not in S_k, are carried into one and the same point of the fixed S_{n-k-1}.*

23. *The units of the quadratic Euclidean domains*, by Professor E. H. Hadlock, University of Florida.

The units of the quadratic Euclidean domains $R(\sqrt{D})$ for $D>0$ are obtained and expressed as integral powers of the fundamental units, where the fundamental unit for a fixed value of D is either the fundamental solution or associated with that solution of one and only one of the equations $x^2-Dy^2=\pm 1, \pm 4$.

24. *Plane arcs which are the union of two similar subarcs*, by Professor M. K. Fort, Jr., University of Georgia.

This paper is concerned with plane arcs A which may be expressed as the union of two subarcs A_1 and A_2 such that: $A_1 \cap A_2$ consists of a single point, A_1 is congruent to A_2, and A is similar to A_1 and A_2. The existence of several different types, and the classification of such arcs is discussed.

25. *A class of analytic finite projective planes*, by Professor J. R. Wesson, Vanderbilt University.

A finite projective plane with $n+1$ points on each line determines a ternary system T with n elements $0, 1, a, b, \cdots$. The system T may be used to construct a projective plane with points $(1, 0, 0)$, $(x, 1, 0)$, and $(x, y, 1)$, where x, y are any elements in T. The lines of the plane have equations $z=0$, $uzy=0$, and $uz(vyx)=0$, where u, v are any elements in T. Any finite projective plane will lead to an analytic projective plane of the same order with points and lines as given above. The analytic plane so determined is analogous to that of Veblen and Wedderburn.

26. *Simplexes on the twisted cubic*, by Mr. Robert Everett, University of Georgia.

In E^3 if a 1-simplex and a 2-simplex have their vertices on the twisted cubic, a curve on which all points are in general position, under what conditions will these simplexes intersect? It is shown by the use of determinants that they intersect only if their vertices alternate along the curve.

27. *Hiroshima and Nagasaki and some of the mathematical problems they raise*, by Professor W. S. Snyder, University of Tennessee. (By invitation.)

To realize the maximum benefits atomic energy has to offer we need to know much more about the long term effects of radiation on man. The populations of the above cities may provide some information and the Atomic Bomb Casualty Commission was organized by the National Research Council to study the long term effects of radiation on the populations of these cities. The interpretation of the data obtained requires a rather accurate estimate of the radiation dose and also poses difficult statistical problems. Some of these are discussed.

28. *A universal metric for the experimental sciences*, by Professor Jose Gallego-Diaz, University of Puerto Rico.

The author asserts that a better description of certain physical phenomena would be obtained by using a metric invariant under a change of scale rather than the Euclidean metric whose characteristic property of invariance under rotation of axes is usually without significance in the representation of such phenomena. With this new Riemannian metric it is found that the curves $y=cx^m$ are geodesics. Extremals for certain integral analogues of Hamilton's principle and the Brachistochrone integral are found. Applications to biology and economics were made.

29. *A discriminant approach to the identification of conic sections*, by Professor F. S. Harper, Georgia State College of Business Administration.

By applying elementary discriminant methods to the general quadratic in x and y, criteria are established for identifying the conic sections, including all of the degenerate cases.

30. *Some new rational distance sets*, by Professor G. B. Huff, University of Georgia.

There is no nonzero rational number x such that $(x, 0)$ is at rational distance from $(0, a)$ and

(a, b) if $a, b = 1, 2$ or $4, 5$. One might conjecture that this statement holds whenever a, b are consecutive integers. On the contrary, it may be shown by a simple elementary argument that if k is an integer such that $k+1$ is a square, there is an infinite number of rational points on the x-axis, each at a rational distance from $(0, 2k)$ and $(0, 2k+1)$. This statement and similar results may be obtained from the properties of a certain elliptic cubic.

31. *The theory of relations*, by Professor Andrew Sobczyk, University of Florida.

A *structure* (in this paper) is any set B with a reflexive and symmetric relation δ. Theorems: *For any structure, there exists a partition $B = \bigcup B_i$ of B into maximal subsets B_i such that $x\delta y$ iff x and y belong to the same B_i. Any structure is a union of metric spaces, for which the distance d assumes only nonnegative integer values, and such that $x\delta y$ iff $d(x, y) = 0$ or 1. For any structure B with a finite number of elements, the partition is unique; in case of an uncountable number of elements, a counter-example shows that the partition is not always unique.* There are many more theorems, examples, and applications.

32. *Sylvester's theorem and the Vandermonde determinant*, by Cadet W. R. Alford, The Citadel.

If one borders the Vandermonde determinant by the column of elements $1, A, A^2, \cdots, A^{n-1}$, $P(A)$ and the row of elements $P(\lambda_L), \cdots, P(\lambda_n), P(A)$, then the determinant set equal to zero is a form of Sylvester's theorem. The λ_i's are the eigenvalues of the matrix A.

33. *Elliptic conic sections*, by Professor D. B. Goodner, The Florida State University.

Elliptic conic sections were studied by analytic methods. Equations for rigid motions of the elliptic plane were developed and used to reduce various second degree equations to standard forms. Systematic study of the standard forms led to the discovery of several basic properties of elliptic conic sections.

34. *Simple proofs of some classical determinantal relations*, by Dr. A. S. Householder, Director, Mathematics Panel, Oak Ridge National Laboratory.

Standard proofs of a number of the classical determinantal relations, both identities and inequalities, generally make use of a cumbersome index notation that is troublesome to write and more troublesome to follow. It is the purpose of this paper to point out that most of this can be dispensed with by a judicious use of partitioned matrices and of permutation matrices. Illustrations will include certain determinantal expansions, the theorem of corresponding minors, theorems on compounds, and a generalized Hadamard inequality.

35. *Present trends in undergraduate college mathematics*—a panel discussion, lead by Professor D. R. Davis, East Carolina College.

The topic was treated as follows: (1) Nature and content of lower division service courses as indicated by recent textbooks and current committees, by Professor Trevor Evans, Emory University. (2) Nature and content of lower division courses for mathematics and science majors as advocated by the Dartmouth and Kansas Summer Writing Groups, the Committee on Undergraduate Program, and current textbooks, by Professor E. A. Cameron, University of North Carolina. (3) Trends in changing content and in new courses in the upper division college curriculum as indicated by recent textbooks and by present trends in the development of both pure and applied mathematics, by Professor F. W. Kokomoor, University of Florida. A summary was given by the panel leader.

36. *Some mathematical aspects of radiation dose calculations*, by Professor W. S. Snyder, University of Tennessee. (By invitation)

If R is a convex phantom irradiated by an external source and if S is any phantom which includes R but does not shield R from the incident radiation then the maximum dose in S is not less

than the maximum dose in R. By similar considerations bounds may often be obtained for maximum dose or for integrated dose in an irradiated phantom as well as for the variation of dose within the phantom. Some isoperimetric problems are posed and discussed briefly.

37. *Some convergence properties of infinite exponentials*, by Professor W. B. Evans, Georgia Institute of Technology.

Analogous to the infinite exponential $E_1^{\infty}a_i$, the infinite exponential $E_{-\infty}^{0}a_i$ and the double infinite exponential $E_{-\infty}^{\infty}a_i$ are defined. Convergence is defined for these exponentials and conditions are specified for the sequence $\{a_i\}$ which will insure convergence.

38. *A generalization of the binary system*, by Dr. Basil Gordon, Army Ordnance Missile Command, Redstone Arsenal, Alabama.

Let $K=\{k_n\}$ be a sequence of positive integers with $k_n<k_{n+1}$. Let S be the set of sums of distinct members of K. If each s in S has a unique representation in such a form, S is called a π-set with base K. Thus the set of all integers is a π-set with base $K=\{2^n\}$. It is shown that if S has measure $m>0$, then $\lim_{n\to\infty} 2^n/k_n=m$ and conversely. The problem of determining $c_n=\min_k k_n$ is also discussed.

39. *A method for the evaluation of certain integrals containing higher transcendental functions*, by Professor M. O. González, University of Alabama.

If $L\{F(t)\}=f(s)$ and $(s+\beta)^{\alpha+1}f(s)=\sum_{n=0}^{\infty} c_n u^n$, where $u=(s-\beta)/(s+\beta)$, then it follows $\int_0 e^{-\beta t}F(t)L^{(\alpha)}(2\beta t)dt=c_n/(2\beta)^{\alpha+1}$ where $L_n^{(\alpha)}$ denotes the generalized Laguerre polynomial. Applications of this result to cases in which $F(t)$ is a higher transcendental function are given.

40. *Some properties of the sub-series of certain series*, by Professor M. G. Boyce, Vanderbilt University.

In a given series if some terms are deleted a sub-series is obtained. Some series have sub-series converging to every number in some interval. Examples and a necessary and sufficient condition are given. Another property possessed by some series is that of representing every rational number of an interval by a sub-series having a finite number of terms. Several general theorems are given, and special series, including $\sum 1/n$ and $\sum 1/n(n+1)$, are discussed.

41. *Combining concepts of Euclid and Dedekind to define the real number system*, by Professor E. B. Shanks, Vanderbilt University.

A geometric description of the analytic definitions employed in the paper follows. A real number is defined as a set of rationals on a finite directed line segment with zero as its only endpoint. Equality is defined as identity. Multiplication is defined by taking as the product of two real numbers the set of products whose two factors come one from each segment. Similar definitions are given for addition, subtraction, and order involving appropriate pairs of real numbers, each containing a positive rational. The definition of addition is then extended by analogy to the rational number system.

42. *On the group of a circulant of order n*, by Professor F. A. Lewis, University of Alabama.

This paper proves that the group of a circulant of order n is $n\phi(n)$ when n is odd and $(n/2)\phi(n)$ when n is even. As a corollary it follows that a circulant of order n is a symmetric function if and only if $n=3$.

43. *A special Pellian equation*, by Professor J. W. Grenier, University of Florida.

Theorems and corollaries are given describing all integral solutions of the equation $\delta x^2-dy^2=2$, with $\delta\cdot d=D$, in terms of $t_n+u_n\sqrt{D}$, solutions of Pell's equation $t^2-Du^2=1$, when t_1 is even and D is a product of distinct odd primes.

44. *Distributive lattices induced by finite groups*, by Dr. H. P. Carter, Oak Ridge National Laboratory.

Dr. Carter defines a maximal join-irreducible element of a lattice L to be a join-irreducible element A such that $A \cap B \neq A$ for all join-irreducible elements B which are distinct from A. Then he proves the theorem: If L is a finite distributive lattice, then a necessary and sufficient condition that L be induced by a finite group is that L contain n distinct maximal join-irreducible elements each connected to 0 by a single chain, and that L contain $(a_1+1) \cdots (a_n+1)$ elements, where $a_1, \cdots, a_n$ are the respective heights of these chains.

45. *A probability associated with the Euclidean algorithm*, by Professor H. S. Thurston, University of Alabama.

Given two integers α and $\beta \neq 0$ of the quadratic field $Ra(\sqrt{m})$ where $m<0$ is square-free. What is the probability that integers μ and ν can be found such that $\alpha=\beta\mu+\nu$ where $0 \leq N(\nu) < N(\beta)$? The paper solves this problem, expressing the probability as a function of m.

H. A. ROBINSON, *Secretary*

CALENDAR OF FUTURE MEETINGS

Fortieth Summer Meeting, University of Utah, Salt Lake City, Utah, August 31–September 3, 1959.

Forty-third Annual Meeting, Conrad Hilton Hotel, Chicago, Illinois, January 28–30, 1960.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN

ILLINOIS

INDIANA

IOWA

KANSAS, Kansas State College of Pittsburg, Spring, 1960.

KENTUCKY, University of Kentucky, Lexington, April, 1960.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, American University, Washington, D. C., December 5, 1959.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March 26, 1960.

MINNESOTA

MISSOURI, Central Missouri State College, Warrensburg, Spring, 1960.

NEBRASKA, University of Nebraska, Lincoln, April 23, 1960.

NEW JERSEY, Princeton University, November 7, 1959.

NORTHEASTERN, Boston College, Chestnut Hill, Massachusetts, November 28, 1959.

NORTHERN CALIFORNIA, University of California, Berkeley, January 16, 1960.

OHIO

OKLAHOMA, Oklahoma City University, October 23, 1959.

PACIFIC NORTHWEST, University of Oregon, Eugene, June 19, 1959.

PHILADELPHIA, University of Delaware, Newark, November 28, 1959.

ROCKY MOUNTAIN

SOUTHEASTERN, University of South Carolina, Columbia, April 1–2, 1960.

SOUTHERN CALIFORNIA, Los Angeles State College, March 12, 1960.

SOUTHWESTERN, Air Force Missile Development Center, Holloman Air Force Base, New Mexico, April, 1960.

TEXAS, San Antonio College, April, 1960.

UPPER NEW YORK STATE, University of Rochester, Spring, 1960.

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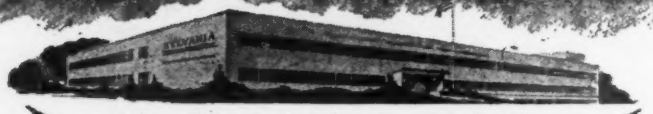

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